Quantum mechanics of an anharmonic oscillator as 0+1 quantum field theory

Classical treatment:
Lagrangian

$$L = \frac{\dot{x}^2}{2} - V(x) = \frac{\dot{x}^2}{2} - \frac{\omega^2}{2}x^2 - \lambda x^4$$

Euler-Lagrange equation ⇔ Newton's 2nd law

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Leftrightarrow \quad \ddot{x} = \omega^2 x + 4\lambda x^3$$

Canonical momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

Hamiltonian

$$H = p\dot{x} - L = \frac{p^2}{2} + \frac{\omega^2}{2}x^2 + \lambda x^4$$

Change of names: $x \to \phi$, $p \to \pi$

$$L = \frac{\dot{\phi}^2}{2} - \frac{\omega^2}{2}\phi^2 - \lambda\phi^4$$

- Lagrangian of the 0+1 scalar field

$$H = \frac{\pi^2}{2} + \frac{\omega^2}{2}\phi^2 + \lambda\phi^4$$

Quantization:

$$\phi \quad \to \quad \widehat{\phi}, \qquad \pi \quad \to \quad \widehat{\pi}$$

Operators ϕ and π act on the wave function $\Phi(\phi)$

$$\widehat{\phi}\Phi(\phi) = \phi\Phi(\phi), \qquad \widehat{\pi}\Phi(\phi) = -i\frac{\partial}{\partial\phi}\Phi(\phi)$$

$[\phi, \pi] = i$ - canonical commutation relation

$$H \to \hat{H} = \frac{\widehat{\pi}^2}{2} + \frac{\omega^2}{2}\widehat{\phi}^2 + \lambda\widehat{\phi}^4$$

Schrodinger picture: $\Phi(\phi, t), \widehat{\phi}, \widehat{\pi}$

Dynamics is governed by the Schrodinger equation

$$i\frac{\partial}{\partial t}\Phi(\phi, t) = \hat{H}\Phi(\phi, t)$$

$\hbar = 1$

Heisenberg picture :

$$
\begin{aligned}
\Phi(\phi) &= \Phi_{\mathrm{Schro}}(\phi, t = 0) \\
\widehat{\phi}(t) &= e^{i\hat{H}t}\widehat{\phi}e^{-i\hat{H}t} \\
\widehat{\pi}(t) &= e^{i\hat{H}t}\widehat{\pi}e^{-i\hat{H}t}
\end{aligned}
$$

Dynamics is determined by Heisenberg equations

$$
\frac{d}{dt}\widehat{\phi}(t) = i[\hat{H}, \widehat{\phi}(t)]
$$

$$
\frac{d}{dt}\pi\widehat{(t)} = i[\hat{H}, \widehat{\pi}(t)]
$$

Perturbation theory $\Leftrightarrow$ expansion in powers of a "coupling constant" λ.

Typical problem: find the dispersion ($\equiv$ mean ϕ^2) in the ground state of the anharmonic oscillator $\langle\Omega|\widehat{\phi}^2|\Omega\rangle$

QM solution:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \quad \hat{H}_0 = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2}\hat{\phi}^2, \quad \hat{H}_{\text{int}} = \lambda\hat{\phi}^4$$

Ground state of the harmonic oscillator ("perturbative vacuum"):

$$\hat{H}_0|0\rangle = E_0|0\rangle \to E_0 = \frac{\omega}{2}, \quad |0\rangle = e^{-\frac{\omega}{2}\phi^2}$$

Ground state of the anharmonic oscillator ("physical vacuum"):

$$\hat{H}|\Omega\rangle = E_{\text{vac}}|\Omega\rangle$$

Perturbative solution

$$|\Omega\rangle = |0\rangle - \lambda \sum{}' |n\rangle \frac{\langle n|\hat{\phi}^4|0\rangle}{E_n - E_0} + O(\lambda^2)$$

$|n\rangle$ - eigenstates of $\hat{H}_0$ (Hermit polynomials), $E_n = \omega(n + \frac{1}{2})$

$$\langle\Omega|\hat{\phi}^2|\Omega\rangle = \langle 0|\hat{\phi}^2|0\rangle - \lambda \sum{}' \frac{\langle 0|\hat{\phi}^2|n\rangle\langle n|\hat{\phi}^4|0\rangle}{E_n - E_0}$$

QFT solution (“interaction picture”)

Some definitions:

Interaction representation:

$$\hat{\phi}_I(t) \equiv e^{i\hat{H}_0 t}\hat{\phi}e^{-i\hat{H}_0 t}$$
$$\hat{\pi}_I(t) \equiv e^{i\hat{H}_0 t}\hat{\pi}e^{-i\hat{H}_0 t}$$

T-product of operators

$$T\{\hat{\phi}(t)\hat{\phi}(t')\} \equiv \theta(t-t')\hat{\phi}(t)\hat{\phi}(t')+\theta(t'-t)\hat{\phi}(t')\hat{\phi}(t)$$

Evolution operator

$$\hat{U}(t,0) \equiv e^{i\hat{H}_0 t}e^{-i\hat{H}t} \quad \Rightarrow \quad \hat{U}^\dagger(t,0) = e^{i\hat{H}t}e^{-i\hat{H}_0 t}$$

$$\hat{U}(t_1,t_2) \equiv U(t_1,0)\hat{U}^\dagger(t_2,0)$$

Group property:

$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) = \hat{U}(t_1,t_3)$$

Formula ($\hat{H}_I(t) \equiv \lambda\hat{\phi}_I^4(t)$):

$$\hat{U}(t,0) = T\exp -i\int_0^t dt'\hat{H}_I(t')$$

$$= 1 - i\int_0^t dt'\hat{H}_I(t') + i^2\int_0^t dt'\int_0^{t'} dt''\,\hat{H}_I(t')\hat{H}_I(t'') + \ldots$$

Proof:

$$\frac{d}{dt}\hat{U}(t,0) = -ie^{i\hat{H}_0 t}\lambda\hat{\phi}^4 e^{-i\hat{H}t} =$$

$$-ie^{i\hat{H}_0 t}\lambda\hat{\phi}^4 e^{-i\hat{H}_0 t}\hat{U}(t,0) = -i\lambda\hat{\phi}_I^4(t)\hat{U}(t,0)$$

$$\frac{d}{dt}(\text{r.h.s.}) = -i\lambda\hat{\phi}_I^4(t)\Big(1 - i\int_0^t dt'\hat{H}_I(t') + \ldots\Big)$$

$$= -i\lambda\hat{\phi}_I^4(t)(\text{r.h.s.})$$

Also, $\hat{U}(t,0)\Big|_{t=0} = (\text{r.h.s.})|_{t=0} = 1$

$\Rightarrow \qquad \hat{U}(t,0) = (\text{r.h.s.})$

Similarly one can prove that

$$\hat{U}(t_1,t_2) = T\exp -i\int_{t_2}^{t_1} dt'\hat{H}_I(t')$$

How to find $\langle\Omega|\hat{\phi}^2|\Omega\rangle$ using the evolution operator?

Key idea: if you take $|0\rangle$ and wait for a long time, you'll get $|\Omega\rangle$.

$$\lim_{T\to\infty(1-i\epsilon)} e^{-i\hat{H}T}|0\rangle = ?$$

(Strictly speaking, we must take $T(1-i\epsilon)$, then $T\to\infty$, and only then $\epsilon\to 0$).

$$\lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|0\rangle =$$
$$\lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}\sum_N |N\rangle\langle N|0\rangle =$$
$$\lim_{T\to\infty} e^{-iE_{\text{vac}}T(1-i\epsilon)}\Big(|\Omega\rangle\langle\Omega|0\rangle +$$
$$\sum{}' |N\rangle\langle N|0\rangle e^{-\epsilon T(E_N-E_{\text{vac}})+iT(E_N-E_{\text{vac}})}\Big)$$

At $T\to\infty$

$e^{-\epsilon T(E_N-E_{\text{vac}})}\to 0$ because $E_N > E_{\text{vac}}$

$$\Rightarrow \lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|0\rangle = |\Omega\rangle\langle\Omega|0\rangle e^{-iE_{\text{vac}}T(1-i\epsilon)}$$

Thus

$$|\Omega\rangle = \lim_{T\to\infty}(e^{-iE_{\text{vac}}T}\langle\Omega|0\rangle)^{-1}e^{-i\hat{H}T}|0\rangle$$

In terms of evolution operators

$$|\Omega\rangle = \lim_{T\to\infty} (e^{-i(E_{\text{vac}}-E_0)T}\langle\Omega|0\rangle)^{-1}\hat{U}(0,-T)|0\rangle$$
$$\langle\Omega| = \lim_{T\to\infty} (e^{-i(E_{\text{vac}}-E_0)T}\langle 0|\Omega|\rangle)^{-1}\langle 0|\hat{U}(T,0)$$

$$\to \langle\Omega|\hat{\phi}^2|\Omega\rangle = \lim_{T\to\infty} \frac{\langle 0|\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T)|0\rangle}{\langle 0|\Omega|\rangle\langle\Omega|0\rangle e^{-2i(E_{\text{vac}}-E_0)T}}$$

($T(1-i\epsilon)$ is always assumed).

By definition of the T-product

$$\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T) = T\{\hat{\phi}_I^2(0)\hat{U}(T,-T)\} =$$
$$T\{\hat{\phi}_I^2(0)\exp -i\int_{-T}^{T} dt'\hat{H}_I(t')\}$$

Also,

$$\langle 0|\hat{U}(T,-T)|0\rangle = \langle 0|e^{i\hat{H}_0 T}e^{-2i\hat{H}T}e^{i\hat{H}_0 T}|0\rangle =$$
$$e^{2iE_0T}\langle 0|e^{-2i\hat{H}T}|0\rangle \to \langle 0|\Omega\rangle\langle\Omega|0\rangle e^{-2i(E_{\text{vac}}-E_0)T}$$

$$\Rightarrow \langle\Omega|\hat{\phi}^2|\Omega\rangle = \lim_{T\to\infty} \frac{\langle 0|T\{\hat{\phi}_I^2(0)e^{-i\int_{-T}^{T}dt'\hat{H}_I(t')}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}dt'\hat{H}_I(t')}\}|0\rangle}$$

At $T = \infty$ we get

$$\langle \Omega | \hat{\phi}^2 | \Omega \rangle = \frac{\langle 0 | T\{\hat{\phi}_I^2(0) e^{-i\int_{-\infty}^{\infty} dt \hat{H}_I(t)}\} | 0 \rangle}{\langle 0 | T\{e^{-i\int_{-\infty}^{\infty} dt \hat{H}_I(t)}\} | 0 \rangle}$$

- master formula for calculations in the interaction representation.

In the first order in perturbation theory

$$e^{-i\int_{-\infty}^{\infty} dt \hat{H}_I(t)} \simeq 1 \ - \ i\lambda \int_{-\infty}^{\infty} dt \hat{\phi}_I^4(t)$$

so

$$\langle \Omega | \hat{\phi}^2 | \Omega \rangle = \langle 0 | \hat{\phi}^2 | 0 \rangle - i\lambda \int_{-\infty}^{\infty} dt \Big[\theta(t) \langle 0 | \hat{\phi}_I^4(t) \hat{\phi}_I^2(0) | 0 \rangle$$
$$+ \theta(-t) \langle 0 | \hat{\phi}_I^2(0) \hat{\phi}_I^4(t) | 0 \rangle - \langle 0 | \hat{\phi}_I^2(0) | 0 \rangle \langle 0 | \hat{\phi}_I^4(0) | 0 \rangle \Big]$$

The correlation functions of the type $\langle 0 | T\{\hat{\phi}_I^2(0) \hat{\phi}_I^4(t)\} | 0 \rangle$ are called Green functions. They are calculated using Feynman rules.

Feynman rules for the Green functions.

Consider the simplest Green function

$$G(t-t') = \langle 0|T\{\widehat{\phi}_I(t)\widehat{\phi}_I(0)\}|0\rangle$$

which is called "propagator".

To find it, we use the ladder operator formalism for the harmonic oscillator:

$$\hat{a} = \frac{\omega\widehat{\phi} + i\widehat{\pi}}{\sqrt{2\omega}}$$

$$\hat{a}^\dagger = \frac{\omega\widehat{\phi} - i\widehat{\pi}}{\sqrt{2\omega}}$$

Properties of ladder operators:

$[\widehat{\phi}, \widehat{\pi}] = i \quad \Rightarrow$

$[\hat{a}, \hat{a}^\dagger] = 1$

– canonical commutation relation in terms of ladder operators.

$\hat{a}|0\rangle = 0$ $\quad\quad$ $\hat{a}$ – "annihilation operator"

$(\hat{a}^\dagger)^n|0\rangle \sim |n\rangle$ $\quad\quad$ $\hat{a}^\dagger$ – "creation operator"

$\widehat{H}_0 = \omega(\widehat{a}^\dagger \widehat{a} + \frac{1}{2})$

Canonical commutation relation $\Rightarrow$
$[\widehat{H}_0, \widehat{a}] = -\omega\widehat{a}$, $[\widehat{H}_0, \widehat{a}^\dagger] = \omega\widehat{a}\dagger \Rightarrow$

$$\left.\begin{array}{l} e^{i\widehat{H}_0 t}\widehat{a}e^{-i\widehat{H}_0 t} = \widehat{a}e^{-i\omega t} \\ e^{i\widehat{H}_0 t}\widehat{a}^\dagger e^{-i\widehat{H}_0 t} = \widehat{a}^\dagger e^{i\omega t} \end{array}\right\} \Rightarrow$$

$$\widehat{\phi}_I(t) = e^{i\widehat{H}_0 t}\widehat{\phi}e^{-i\widehat{H}_0 t} =$$

$$\frac{1}{\sqrt{2\omega}} e^{i\widehat{H}_0 t}(\widehat{a} + \widehat{a}^\dagger)e^{-i\widehat{H}_0 t} = \frac{\widehat{a}}{\sqrt{2\omega}}e^{-i\omega t} + \frac{\widehat{a}^\dagger}{\sqrt{2\omega}}e^{i\omega t}$$

Now we can find the "propagator"

$$\langle 0|T\{\widehat{\phi}_I(t)\widehat{\phi}_I(t')\}|0\rangle =$$

$$\frac{\theta(t-t')}{2\omega}\langle 0|(\widehat{a}e^{-i\omega t} + \widehat{a}^\dagger e^{i\omega t})(\widehat{a}e^{-i\omega t'} + \widehat{a}^\dagger e^{i\omega t'})|0\rangle$$

$$+(t \leftrightarrow t') =$$

$$\frac{1}{2\omega}\langle 0|[\widehat{a}, \widehat{a}^\dagger]e^{-i\omega(t-t')}|0\rangle + (t \leftrightarrow t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}$$

(recall that $\widehat{a}|0\rangle = \langle 0|\widehat{a}^\dagger = 0$).
Similarly

$$\langle 0|T\{\widehat{\phi}_I^2(0)\widehat{\phi}_I^4(t)\}|0\rangle =$$

$$\frac{\theta(t)}{8\omega^3}\langle 0|(\widehat{a}e^{-i\omega t} + \widehat{a}^\dagger e^{i\omega t})^4(\widehat{a} + \widehat{a}^\dagger)^2|0\rangle + (t \leftrightarrow t')$$

The result of the calculation can be represented by Wick's theorem:

$$\langle 0|T\{\hat{\phi}_I^2(0)\hat{\phi}_I^4(t)\}|0\rangle =$$

$$\sum_{\text{contractions}} \hat{\phi}_I(0)\hat{\phi}_I(0)\hat{\phi}_I(t)\hat{\phi}_I(t)\hat{\phi}_I(t)\hat{\phi}_I(t)$$

Each contraction is a propagator

$$\hat{\phi}_I(t)\hat{\phi}_I(0) = G(t) = \frac{1}{2\omega}e^{-i\omega|t|}$$

represented by a line in a Feynman diagram

The rest is combinatorics

$$\langle 0|T\{\hat{\phi}_I^2(0)\hat{\phi}_I^4(t)\}|0\rangle =$$

$$\langle 0|\hat{\phi}_I^2(0)|0\rangle\langle 0|\hat{\phi}_I^4(0)\}|0\rangle =$$

$$\Rightarrow \langle 0|T\{\hat{\phi}_I^2(0)\hat{\phi}_I^4(t)\}|0\rangle - \langle 0|\hat{\phi}_I^2(0)|0\rangle\langle 0|\hat{\phi}_I^4(0)\}|0\rangle =$$

$= \sum$ of the connected Feynman diagrams =

$$= \quad 12G^2(t)G(0)$$

Second term (coming from the denominator) cancels disconnected diagrams. This is a general property: any Green function is represented by sum of relevant connected Feynman diagrams.

$$\Rightarrow \langle \Omega|\hat{\phi}^2|\Omega\rangle = G(0) - 12i\lambda\int_{-\infty}^{\infty} dt G^2(t)G(0)$$

$$= \frac{1}{\omega^2}(1 - \frac{3\lambda}{2\omega^3} i\int_{-\infty}^{\infty} dt e^{-2i\omega|t|}) =$$

$$\frac{1}{4\omega^2}(1 - \frac{6\lambda}{\omega^2}) + O(\lambda^2)$$

This may be a wierd way to calculate $\langle\phi^2\rangle$ in quantum mechanics, but it generalizes to field theories.

QFT for the Klein-Gordon field

$\phi(\vec{x},t)$ – Klein-Gordon field.
(if $m_\pi = 0$ it would be observable like electric field).
$(\partial^2 - m^2)\phi(x) = 0$ – Klein-Gordon equation
– analog of Maxwell's equations.
$x \equiv (\vec{x},t)$, $\partial^2 \equiv \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}$

Classical theory:
Lagrangian $L = \int d^3x \mathcal{L}(\vec{x},t)$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

– Lagrangian density for the free KG field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \lambda\phi^4$$

– Lagrangian density for the self-interacting KG field .
Euler-Lagrange eqn reproduces the KG equation

Quantization of the free KG field

For simplicity: one-dimensional KG field $\phi(x, t)$

$$L = \int dx \left(\frac{\dot{\phi}^2}{2} - \frac{(\phi')^2}{2} - \frac{m^2}{2}\phi^2 \right)$$

Idea: KG field $\Leftrightarrow$ superposition of harmonic oscillators.

Lattice model of the KG field:

- harmonic oscillators at each point of the lattice with pairwise interaction.

$$L(t) = a \sum \left[\frac{\dot{\phi}_n^2(t)}{2} - \frac{(\phi_{n+1}(t) - \phi_n(t))^2}{2a} - \frac{m^2}{2}\phi_n^2(t) \right]$$

Change of the label: $\phi_n \to \phi(x_n, t) \Rightarrow$

$$L(t) = a \sum \left[\frac{\dot{\phi}^2(x_n,t)}{2} - \frac{(\phi(x_n+a,t) - \phi(x_n,t))^2}{2a^2} - \frac{m^2}{2}\phi^2(x_n, t) \right]$$

In the "continuum limit" $a \to 0$ this reproduces the above KG Lagrangian

Quantization of a set of oscillators.

Canonical momenta: $\pi_n = \frac{\partial L}{\partial \dot\phi_n} = a\dot\phi_n$

Define $\pi(x_n,t) \equiv \frac{1}{a}\pi_n = \dot\phi(x_n,t) \Rightarrow$

$H = \sum \pi_n \dot\phi_n - L =$

$a\sum\left[\frac{\pi^2(x_n,t)}{2} + \frac{(\phi(x_n+a,t)-\phi(x_n,t))^2}{2a^2} + \frac{m^2}{2}\phi^2(x_n,t)\right]$

In the continuum limit we get

$$H = \int dx\Big[\frac{\pi^2(x,t)}{2} + \frac{\phi'(x,t)^2}{2} + \frac{m^2}{2}\phi^2(x,t)\Big]$$

As usual, for quantization of the set of oscillators we promote ϕ_n and π_n to operators satisfying canonical commutation relations

$$[\hat\phi_m, \hat\pi_n] = i\delta_{mn}, \quad [\hat\phi_m, \hat\phi_n] = [\hat\pi_m, \hat\pi_n] = 0$$

In terms of $\phi(x_n)$ and $\pi(x_n)$ this reads

$$[\hat\phi(x_m), \hat\pi(x_n)] = \frac{i}{a}\delta_{mn}$$

which reduces to

$$[\hat\phi(x), \hat\pi(y)] = i\delta(x-y),$$
$$[\hat\phi(x), \hat\phi(y)] = [\hat\pi(x), \hat\pi(y)] = 0$$

in the continuum limit.

For the 3-dimensional KG field the canonical commutation relations are

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^3(x-y),$$
$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0$$

Ladder operators

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}}\left(\hat{a}_{\vec{p}} e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}\right)$$
$$\hat{\pi}(\vec{x}) = -i\int \frac{d^3p}{\sqrt{2E_p}} E_p\left(\hat{a}_{\vec{p}} e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}\right)$$

$(E_p = \sqrt{m^2 + \vec{p}^2})$.

It is easy to check that

$$\left.\begin{array}{l} \left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = (2\pi)^3\delta^3(\vec{p}-\vec{p}') \\ \left[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}\right] = \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ canonical commutation relations.

Classical Hamiltonian

$$H = \int d^3x \Big[\frac{\pi^2(\vec{x},t)}{2} + \frac{\vec{\nabla}\phi(\vec{x},t)^2}{2} + \frac{m^2}{2}\phi^2(\vec{x},t)\Big]$$

$$\Rightarrow \hat{H} = \int d^3x \Big[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x})\Big]$$

In terms of ladder operators

$$\hat{H} = \int d^3p E_p \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}$$

(we throw away the ∞ constant).

$\Rightarrow [\hat{H}, \hat{a}_{\vec{p}}] = -E_p \hat{a}_{\vec{p}}$, $[\hat{H}, \hat{a}^\dagger_{\vec{p}}] = E_p \hat{a}^\dagger_{\vec{p}}$. $\Rightarrow$

$$e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t} = \hat{a}_{\vec{p}}e^{-iE_pt}, \quad e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t} = \hat{a}^\dagger_{\vec{p}}e^{iE_pt}$$

The Heisenberg operators are defined as usual

$$\hat{\phi}(\vec{x},t) = e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}, \quad \hat{\pi}(\vec{x},t) = e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t}$$

$$\Rightarrow \quad \begin{array}{l} \hat{\phi}(x) = \int \frac{d^3p}{\sqrt{2E_p}}\Big(\hat{a}_{\vec{p}}e^{-ipx} + \hat{a}^\dagger_{\vec{p}}e^{ipx}\Big) \\ \hat{\pi}(x) = -i\int \frac{d^3p}{\sqrt{2E_p}}E_p\Big(\hat{a}_{\vec{p}}e^{-ipx} - \hat{a}^\dagger_{\vec{p}}e^{ipx}\Big) \end{array}$$

where $x = (\vec{x},t)$ and $px = E_pt - \vec{p}\vec{x}$

Basic property

$$\hat{a}_{\vec{p}}|0\rangle = 0$$

where $|0\rangle$ is the ground state of the quantized field ($\equiv$ the lattice of oscillators).

Proof:

Suppose $\hat{a}_{\vec{p}}|0\rangle \neq 0$. Denote this state by $|X\rangle$.

$$\hat{H}|X\rangle = \hat{H}\hat{a}_{\vec{p}}|0\rangle = [\hat{H}, \hat{a}_{\vec{p}}]|0\rangle + \hat{a}_{\vec{p}}H|0\rangle =$$
$$-E_p\hat{a}_{\vec{p}}|0\rangle + E_0\hat{a}_{\vec{p}}E_0|0\rangle = (E_0 - E_p)|X\rangle$$

We see, that the state $|X\rangle$ has enegry less than the ground state energy which is impossible $\Rightarrow \hat{a}_{\vec{p}}|0\rangle = 0$.

Free propagator $G_0(x-y) \equiv \langle 0|T\{\hat{\phi}_I(x)\hat{\phi}_I(y)\}|0\rangle$

$$G_0(x-y) =$$
$$\theta(x_0-y_0)\langle 0| \int \frac{d^3p}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}}e^{-ipx} + \hat{a}^\dagger_{\vec{p}}e^{ipx}\right)$$
$$\int \frac{d^3p'}{\sqrt{2E'_p}} \left(\hat{a}_{\vec{p}'}e^{-ip'y} + \hat{a}^\dagger_{\vec{p}'}e^{+ip'y}\right)|0\rangle + (x \leftrightarrow y)$$

$\hat{a}|0\rangle = \langle 0|\hat{a}^\dagger = 0 \Rightarrow$

$$G_0(x-y) =$$
$$\theta(x_0-y_0) \int \frac{d^3p d^3p'}{2\sqrt{E_p E'_p}} e^{-ip(x-y)} \langle 0|\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}'}|0\rangle + (x \leftrightarrow y)$$

$[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = \delta^3(\vec{p}-\vec{p}') \quad \Rightarrow$

$$G_0(x-y) =$$
$$\theta(x_0-y_0) \int \frac{d^3p}{2E_p} e^{-iE_p(x-y)_0 + i\vec{p}(\vec{x}-\vec{y})} + (x \leftrightarrow y)$$

It can be rewritten in the rel.-inv. form:

$$G_0(x-y) = \lim_{\epsilon\to 0} \int \frac{d^4p}{16\pi^4 i} e^{-ip(x-y)} \frac{1}{m^2-p^2-i\epsilon}$$

Self-interacting Klein-Gordon field

$$L = \int d^3x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \lambda\phi^4$$

$$H = \int d^3x \Big[\frac{\pi^2(\vec{x},t)}{2} + \frac{\vec{\nabla}\phi(\vec{x},t)^2}{2} + \frac{m^2}{2}\phi^2(\vec{x},t) + \lambda\phi^4\Big]$$

Quantization - same as for the free KG field:
$\phi \to \hat{\phi}(\vec{x})$, $\pi \to \hat{\pi}(\vec{x})$

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \delta^3(\vec{x}-\vec{y}), \qquad [\hat{\phi},\hat{\phi}] = [\hat{\pi},\hat{\pi}] = 0$$

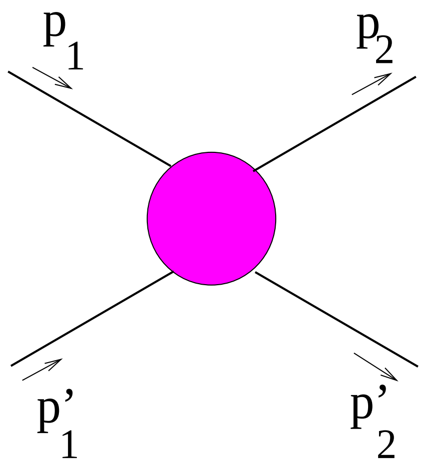
$$\hat{H} = \int d^3x \Big[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x}) + \lambda\hat{\phi}^4\Big]$$

Heisenberg picture: $\hat{\phi}(x)$ and $\hat{\pi}(x)$ depend on time

$$\hat{\phi}(x) \equiv \hat{\phi}(\vec{x},t) = e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}$$
$$\hat{\pi}(x) \equiv \hat{\pi}(\vec{x},t) = e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t}$$

Vectors of state (like the vector of ground state $|\Omega\rangle$) do not depend on time.

Cross sections are determined by Green functions $\langle\Omega|T\{\widehat{\phi}(x_1)\widehat{\phi}(x_2)...\widehat{\phi}(x_n)\}|\Omega|\rangle$



$$\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{1}{64\pi^2 s}|T|^2$$

$$s=(p_1+p'_1)^2= 4E^2_{c.m.}$$

$$S(p_1,p'_1 \to p_2,p'_2) = 1+\delta(\sum p_1 - \sum p_2)T(p_1,p'_1 \to p_2,p'_2)$$

LSZ theorem:

$$S(p_1,p'_1 \to p_2,p'_2) = \lim_{p_i^2 \to m^2} \Pi(p_i^2 - m^2) \int d^4x^4_1x'_1d^4x^4_2x'_2e^{-ip_1x_1-ip'_1x'_1+ip_2x_2-ip'_2x'_2} \langle\Omega|T\{\widehat{\phi}(x_1)\widehat{\phi}(x'_1)\widehat{\phi}(x_2)\widehat{\phi}(x'_2)\}|\Omega\rangle$$

Perturbation theory: $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$

$$\hat{H}_0 = \int d^3x \Big[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x})\Big]$$
$$\hat{H}_{\text{int}} = \int d^3x \lambda \hat{\phi}^4(\vec{x})$$

Interaction representation defined as in QM

$$\hat{\phi}_I(\vec{x},t) \equiv e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}, \ \hat{\pi}_I(\vec{x},t) \equiv e^{i\hat{H}_0 t}\hat{\pi}(\vec{x})e^{-i\hat{H}_0 t}$$

Literally repeating all the steps we get

$$\langle\Omega|\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\langle 0|T\{\hat{\phi}_I(x_1)...\hat{\phi}_I(x_n)e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|0\rangle}$$

where $|0\rangle$ is the "perturbative vacuum" ($\equiv$ ground state of $\hat{H}_0$).

$\int_{-\infty}^{\infty} dt\hat{H}_I(t) = -\int d^4x\hat{\mathcal{L}}_I(x) \quad \Rightarrow$

$$\langle\Omega|\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\langle 0|T\{\hat{\phi}_I(x_1)...\hat{\phi}_I(x_n)e^{i\int d^4x\hat{\mathcal{L}}_I(x)}\}|0\rangle}{\langle 0|T\{e^{i\int d^4x\hat{\mathcal{L}}_I(x)}\}|0\rangle}$$

This master formula is relativistic invariant (although the intermediate steps were not).

Cross section in the first order in λ.

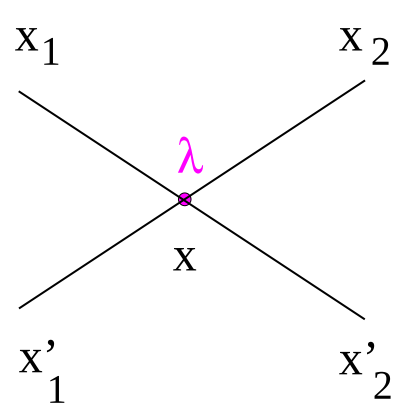
$$\langle 0|T\{\hat{\phi}_I(x_1)\hat{\phi}_I(x'_1)\hat{\phi}_I(x_1)\hat{\phi}_I(x'_2)i\lambda\int d^4x\hat{\phi}_I^4(x)\}|0\rangle$$
$$= \sum_{\text{contractions}}$$

$$\hat{\phi}_I(x_1)\hat{\phi}_I(x'_1)\hat{\phi}_I(x_1)\hat{\phi}_I(x'_2)i\lambda\int d^4x\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)$$

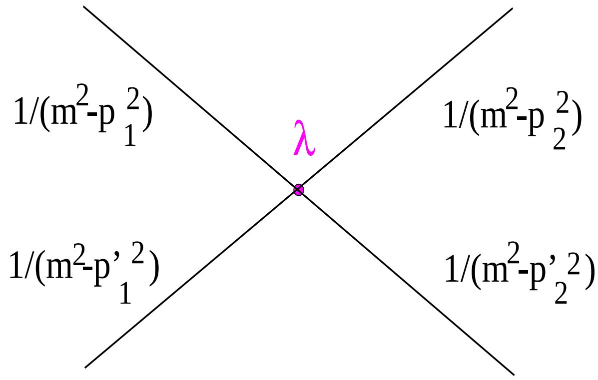
Each contraction is a free propagator

$$\hat{\phi}_I(x_1)\hat{\phi}_I(x) = G_0(x_1 - x)$$

$\Rightarrow$ Feynman diagram for the four-point Green function



Feynman diagram in the momentum representation



$$G(p_1, p_1' \to p_2, p_2') = \int d^4x_1^4x_1' d^4x_2^4x_2' e^{-ip_1x_1 - ip_1'x_1' + ip_2x_2 - ip_2'x_2'}$$
$$\langle\Omega|T\{\hat{\phi}(x_1)\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_2')\}|\Omega\rangle = \frac{24\lambda}{(m^2-p_1^2)(m^2-p_1'^2)(m^2-p_2^2)(m^2-p_2'^2)} + O(\lambda^2)$$

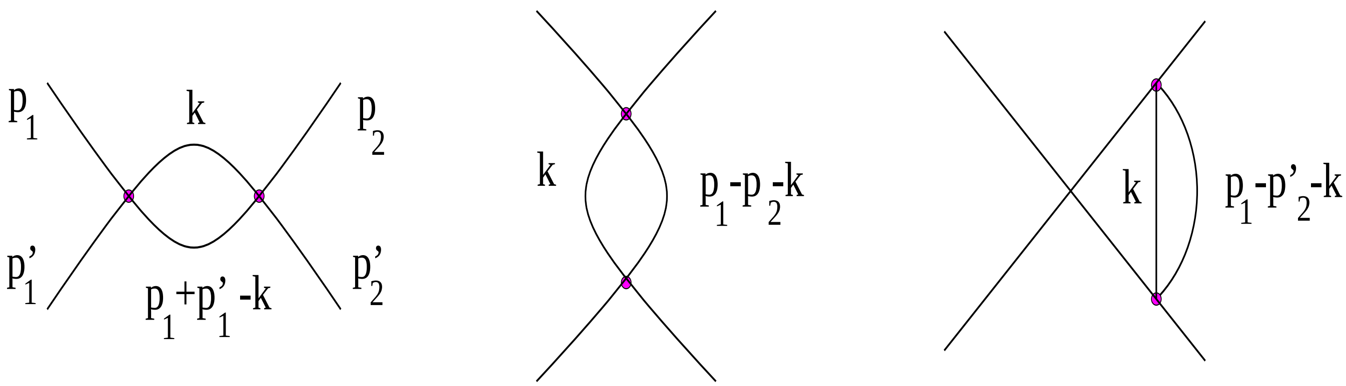
LSZ theorem $\Rightarrow$

$$T(p_1, p_1' \to p_2, p_2') = 24\lambda + O(\lambda^2) \Rightarrow$$

The meson-meson cross section is

$$\frac{d\sigma}{d\Omega} = \frac{9\lambda^2}{\pi^2 s} + O(\lambda^4)$$

In higher orders in perturbative expansion in powers of λ we have more complicated diagrams such as



Feynman rules for the Green functions in momentum representation:

- $\frac{1}{m^2-p^2-i\epsilon}$ for each propagator with momentum p.
- 24λ for each vertex (24- combinatorial coefficient)
- $\int \frac{d^4k}{16\pi^4 i}$ for each "loop" momentum k.

Path integrals in quantum mechanics

Anharmonic oscillator

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{p}^2}{2}, \hat{V} = \frac{\omega^2}{2}\hat{x}^2 + \lambda\hat{x}^4$$

(note that here $\hat{H}_0$ does not include $\frac{\omega^2}{2}\hat{x}^2$).
Wave function in Dirac notations is $\Psi(x,t) = \langle\phi|\Psi(t)\rangle$
where $|\phi\rangle$- eigenstates of the coordinate operator $\hat{x}$: $\hat{x}|x\rangle = x|x\rangle$.
Evolution is described by the operator $e^{-i\hat{H}t}$:

$$|\Psi(t_f)\rangle = e^{-i\hat{H}(t_f-t_i)}|\Psi(t_i)\rangle$$

In terms of wave functions

$$\Psi(x_f,t_f) = \langle x_f|\Psi(t)\rangle = \int dx_i \langle x_f|e^{-i\hat{H}(t_f-t_i)}|x_i\rangle\Psi(x_i,t_i)$$

$\Rightarrow K(x_f,t_f;x_i,t_i) = \langle x_f|e^{-i\hat{H}(t_f-t_i)}|x_i\rangle$ - kernel of the evolution operator (propagation amplitude).
Physical meaning: we release the oscillating particle in the point $x = x_i$ at time $t = t_i$ and $K(x_f,t_f;x_i,t_i)$ is a probabilty to find this particle in the point $x = x_f$ at later time $t = t_f$.

Path integral for the evolution kernel

Insert $1 = \int dx |x\rangle\langle x|$ n times ($\Delta t = \frac{t_f - t_i)}{n}$)

$$\langle x_f | e^{-i\widehat{H}(t_f - t_i)} | x_i \rangle =$$

$$\int dx_1 ... \int dx_n \langle x_f | e^{-i\widehat{H}\Delta t} | x_n \rangle ... \langle x_2 | e^{-i\widehat{H}\Delta t} | x_1 \rangle \langle x_1 | e^{-i\widehat{H}\Delta t} | x_i \rangle$$

For small Δt

$$\langle x_k | e^{-i\widehat{H}\Delta t} | x_{k-1} \rangle =$$

$$\int \frac{dp}{2\pi} \langle x_k | p \rangle \langle p | e^{-i\widehat{H}\Delta t} | x_{k-1} \rangle =$$

$$\int \frac{dp}{2\pi} e^{ip(x_k - x_{k-1})} e^{-i\frac{p^2}{2}\Delta t - iV(x_k)\Delta t} =$$

$$\frac{1}{\sqrt{2\pi i \Delta t}} e^{i\frac{(x_k - x_{k-1})^2}{2\Delta t} - iV(x_k)\Delta t}$$

$$\langle x_f | e^{-i\widehat{H}(t_f - t_i)} | x_i \rangle =$$

$$\left(\frac{1}{2\pi i \Delta t}\right)^{n/2} \int dx_1 ... \int dx_n e^{i\Delta t \sum \left[\frac{(x_k - x_{k-1})^2}{2(\Delta t)^2} - V(x_k)\right]}$$

$$\Rightarrow N^{-1} \int Dx(t) e^{i\int_{t_i}^{t_f} dt \left(\frac{\dot{x}^2}{2} - V(x)\right)}$$

Path integral and classical mechanics

Restore $\hbar$ for a minute

$$K(x_f,t_f;x_i,t_i) \;=\; N^{-1}\int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t)e^{\frac{i}{\hbar}S(x(t))}$$

This formula can be used as a postulate of quantum mechanics instead of Schrodinger equation.

Classical limit.

At $\hbar \to 0$ (classical limit) this integral is determined by a stationary phase point corresponding to minimum of the action $S(x(t)) \Rightarrow$ least action principle - given the initial and final points, the classical path is a path with minimal action.

In quantum mechanics, all trajectories are possible. Each trial path is weighted with e^{iS} and we have to sum over trial configurations due to the superposition principle of quantum mechanics (for undistinguishable paths, we must sum the amplitudes).

Path integrals for the Green functions

Consider the two-point Green function

$$G(t_1, t_2) = \langle \Omega | T\{\hat{\phi}(t_1)\hat{\phi}(t_2)\} | \Omega \rangle$$

At first, we prove that ($t_{fi} \equiv t_f - t_i$)

$$G(t_1, t_2) = \lim_{t_f \to \infty, t_i \to -\infty} \frac{\langle 0 | T\{e^{-i\hat{H}t_{fi}} \hat{\phi}(t_1)\hat{\phi}(t_2)\} | 0 \rangle}{\langle 0 | e^{-i\hat{H}t_{fi}} | 0 \rangle}$$

Proof:

Take $t_1 > t_2$. Consider the numerator

$$\mathrm{Num}(t_1, t_2) = \langle 0 | e^{-i\hat{H}t_{f1}} \hat{\phi} e^{-i\hat{H}t_{12}} \hat{\phi} e^{-i\hat{H}t_{2i}} | 0 \rangle$$

Recall that

$$\left.\begin{array}{l} e^{-i\hat{H}t_{2i}}|0\rangle \stackrel{t_i \to -\infty}{\to} e^{-iE_{\mathrm{vac}}t_{2i}} |\Omega\rangle\langle\Omega|0\rangle \\ \langle 0| e^{-i\hat{H}t_{f1}} \stackrel{t_f \to \infty}{\Rightarrow} \langle 0|\Omega\rangle\langle\Omega| e^{-iE_{\mathrm{vac}}t_{f1}} \end{array}\right\} \Rightarrow$$

$$\lim_{t_f \to \infty, t_i \to -\infty} \mathrm{Num}(t_1, t_2) =$$
$$|\langle 0|\Omega\rangle|^2 e^{-iE_{\mathrm{vac}}(t_{f1}+t_{2i})} \langle \Omega | \hat{\phi} e^{i\hat{H}t_{12}} \hat{\phi} | \Omega \rangle$$

Similarly, the denominator is

$$\text{Den} = \langle 0|e^{-i\hat{H}t_{fi}}|0\rangle \Rightarrow |\langle 0|\Omega\rangle|^2 e^{-iE_{\text{vac}}(t_{fi})}$$

$$\Rightarrow \frac{\text{Num}}{\text{Den}} = e^{iE_{\text{vac}}(t_{12})}\langle\Omega|\hat{\phi}e^{-i\hat{H}t_{12}}\hat{\phi}|\Omega\rangle$$

On the other hand

$$\langle\Omega|e^{i\hat{H}t_1}\hat{\phi}e^{-i\hat{H}t_{12}}\hat{\phi}e^{-i\hat{H}t_2}|\Omega\rangle = e^{iE_{\text{vac}}t_{12}}\langle\Omega|\hat{\phi}e^{i\hat{H}t_{12}}\hat{\phi}|\Omega\rangle$$

$$\Rightarrow \lim_{t_f\to\infty, t_i\to-\infty} \frac{\text{Num}}{\text{Den}} = G(t_1, t_2)$$

Path integral for $G(t_1, t_2)$.

We know that

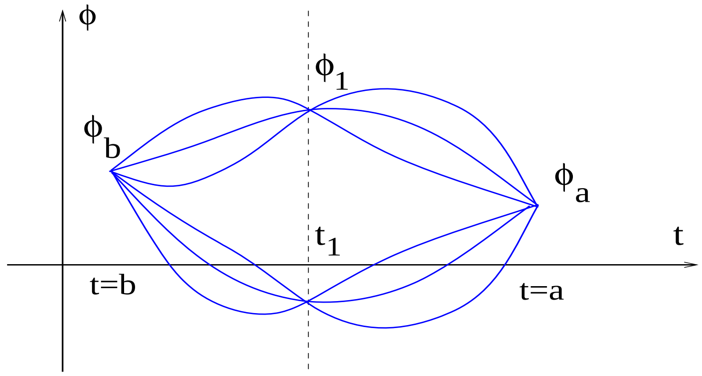
$$\text{Den} = \langle 0|e^{-i\hat{H}t_{fi}}|0\rangle = \int D\phi e^{i\int_{t_i}^{t_f} L(t)dt}$$

The numerator

$$\text{Num} = \langle 0|e^{-i\hat{H}t_{f1}}\hat{\phi}e^{-i\hat{H}t_{12}}\hat{\phi}e^{-i\hat{H}t_{2i}}|0\rangle =$$

$$\int d\phi_1 d\phi_2 \langle 0|e^{-i\hat{H}t_{f1}}\hat{\phi}|\phi_1\rangle\langle\phi_1|e^{-i\hat{H}t_{12}}\hat{\phi}|\phi_2\rangle\langle\phi_2|e^{-i\hat{H}t_{2i}}|0\rangle$$

$$\int d\phi_1 d\phi_2 \langle 0|e^{-i\hat{H}t_{f1}}|\phi_1\rangle\phi_1\langle\phi_1|e^{-i\hat{H}t_{12}}|\phi_2\rangle\phi_2\langle\phi_2|e^{-i\hat{H}t_{2i}}|0\rangle$$



Each path is weighted with $F(\phi_1) = F(\phi(t_1))$

Property

$$\int d\phi_1 \int_{\phi(t_1)=\phi_1}^{\phi(a)=\phi_a} D\phi e^{i\int_{t_1}^{a} L(t)dt} F(\phi_1)$$

$$\int_{\phi(b)=\phi_b}^{\phi(t_1)=\phi_1} D\phi e^{i\int_{b}^{t_1} L(t)dt}$$

$$= \int_{\phi(b)=\phi_b}^{\phi(a)=\phi_a} D\phi e^{i\int_{b}^{a} L(t)dt} F(\phi(t_1)) \qquad \Rightarrow$$

$$\text{Num} = \int D\phi\phi(t_1)\phi(t_2)e^{i\int_{t_i}^{t_f} L(t)dt}$$

$$\Rightarrow G(t_1, t_2) = \frac{\int D\phi\phi(t_1)\phi(t_2)e^{i\int_{-\infty}^{\infty} L(t)dt}}{\int D\phi e^{i\int_{-\infty}^{\infty} L(t)dt}}$$

In general

$$\langle\Omega|T\{\phi(t_1)...\phi(t_n)\}|\Omega\rangle = \frac{\int D\phi\phi(t_1)...\phi(t_n)e^{iS(\phi)}}{\int D\phi e^{iS(\phi)}}$$

Euclidean path integral

Analytic continuation of the evolution operator to the imaginary time $\tau = -it$ gives

$$\langle x_f | e^{-\hat{H}(\tau_f - \tau_i)} | x_i \rangle = N^{-1} \int Dx(\tau) e^{-\int_{\tau_i}^{\tau_f} dt \left(\frac{\dot{x}^2(\tau)}{2} + V(x(\tau)) \right)}$$

($\dot{x}(\tau)$ denotes now the derivative with respect to τ.)

The finite-sum version of this formula

$$\langle x_f | e^{-\hat{H}(\tau_f - \tau_i)} | x_i \rangle =$$

$$\left(\frac{1}{2\pi i \Delta t} \right)^{n/2} \int dx_1 ... \int dx_n e^{-\Delta t \sum \left[\frac{(x_k - x_{k-1})^2}{2(\Delta t)^2} + V(x_k) \right]}$$

is very convenient for practical calculations since the integrals of this type can be computed using the Monte-Carlo methods. This is the simplest example of the lattice calculation in a quantum theory.

Feynman diagams from path integrals

Consider the two-point Green function

$$\langle\Omega|T\{\phi(t_1)\phi(t_2)\}|\Omega\rangle = \frac{\int D\phi\phi(t_1)\phi(t_2)e^{iS(\phi)}}{\int D\phi e^{iS(\phi)}}$$

for the anharmonic oscillator.
At small λ

$$S = S_0 + S_{\text{int}}$$
$$S_0 = \int dt\Big[\frac{\dot\phi^2(t)}{2} - \frac{\omega^2}{2}\phi^2(t)\Big]$$
$$S_{\text{int}} = \int dt\lambda\phi^4(t)$$

Note that, unlike p.28, we include $\frac{\omega^2}{2}\phi^2$ in S_0.

$$\langle\Omega|T\{\phi(t_1)\phi(t_2)\}|\Omega\rangle = \frac{\int D\phi\ \phi(t_1)\phi(t_2)e^{iS_0(\phi)+iS_{\text{int}}(\phi)}}{\int D\phi e^{iS_0(\phi)+iS_{\text{int}}(\phi)}}$$

Perturbative expansion $\Leftrightarrow$ expansion of $e^{iS_{\text{int}}}$ in the numerator and denominator.

In the first order in perturbation theory we get

$$i\lambda \frac{\int D\phi\phi(t_1)\phi(t_2)\int dt\phi^4(t)e^{iS_0}}{\int D\phi e^{iS_0}}$$
$$-i\lambda \frac{\int D\phi\phi(t_1)\phi(t_2)e^{iS_0}}{\int D\phi e^{iS_0}} \quad \frac{\int D\phi \int dt\phi^4(t)}{\int D\phi e^{iS_0}}$$

It is the same expansion as in interaction representation picture (see p.9):

$$\langle\Omega|T\{\hat{\phi}(t_1)\hat{\phi}(t_2)\}|\Omega\rangle = \langle 0|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\}|0\rangle -$$
$$i\lambda\int dt\Big[\langle 0|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\hat{\phi}_I^4(t)\}|0\rangle$$
$$-\langle 0|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\}|0\rangle\langle 0|\hat{\phi}_I^4(t)|0\rangle\Big]$$

because

$$\frac{\int D\phi\phi(t_1)...\phi(t_n)e^{iS_0}}{\int D\phi e^{iS_0}} = \langle 0|T\{\hat{\phi}_I(t_1)...\hat{\phi}_I(t_n)\}|0\rangle$$

Proof: this is simply the master formula for path-integral representation of Green functions (see p.32) applied for the unperturbed harmonic oscillator. (Recall that $\hat{\phi}_I$ is a Heisenberg operator for $\hat{H} = \hat{H}_0$).

Path-integral derivation of Wick's theorem

$$G_0(t_1, ...t_n) = \frac{\int D\phi\phi(t_1)...\phi(t_n)e^{iS_0}}{\int D\phi e^{iS_0}} =$$

$$\sum_{\text{contractions}} = G_0(t_1 - t_2)G_0(t_3 - t_4)...G_0(t_{n-1} - t_n)$$

Define the functional

$$Z(J) = \frac{\int D\phi e^{iS_0 + i\int dt J(t)\phi(t)}}{\int D\phi e^{iS_0}}$$

Expansion of this generating functional $Z(J)$ in powers of the source $J(t)$ generates the set of Green functions G_0:

$$Z(J) =$$

$$1 + i\int dt G_0(t)J(t) + \frac{i^2}{2}\int dt_1 dt_2 J(t_1)J(t_2)G_0(t_1, t_2)$$

$$\frac{i^3}{3!}\int dt_1 dt_2 dt_3 J(t_1)J(t_2)J(t_3)G_0(t_1, t_2, t_3) + ...$$

The generating functional $Z(J)$ is a gaussian integral (albeit a path one) so we can try to calculate by appropriate shift of variable.

Accurately

$$Z(J) = \frac{\int d\phi_f d\phi_i e^{-\frac{\omega}{2}\phi_f^2} e^{-\frac{\omega}{2}\phi_i^2} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS_0 + i\int_{t_i}^{t_f} dt J(t)\phi(t)}}{\int d\phi_f d\phi_i e^{-\frac{\omega}{2}\phi_f^2} e^{-\frac{\omega}{2}\phi_i^2} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS_0}}$$

Let us make a shift $\phi(t) \to \phi(t) + \bar{\phi}(t)$ in the path integral in the numerator. The exponential in the numerator will turn to

$$-\frac{\omega}{2}(\phi_f + \bar{\phi}_f)^2 - \frac{\omega}{2}(\phi_i + \bar{\phi}_i)^2 + iS_0(\phi + \bar{\phi}) +$$
$$i\int_{t_i}^{t_f} dt J(t)(\phi(t) + \bar{\phi}(t)) =$$
$$-\frac{\omega}{2}\bar{\phi}_f^2 - \frac{\omega}{2}\bar{\phi}_i^2 + iS_0(\bar{\phi}) + i\int_{t_i}^{t_f} dt J(t)\bar{\phi}(t)$$
$$-\omega(\phi_f\bar{\phi}_f - \phi_i\bar{\phi}_i) + i\int_{t_i}^{t_f} dt(\dot{\phi}\dot{\bar{\phi}} - \omega^2\phi\bar{\phi})$$
$$-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi) + i\int_{t_i}^{t_f} dt J(t)(\phi(t)) =$$

$$\int_{t_i}^{t_f} dt(\dot{\phi}\dot{\bar{\phi}} - \omega^2\phi\bar{\phi}) \Rightarrow \text{by parts} \Rightarrow$$
$$\dot{\bar{\phi}}\phi\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt\phi[(-\partial_t^2 + \omega^2)\bar{\phi}(t) + J(t)] \quad \Rightarrow$$

$$= -\frac{\omega}{2}\bar{\phi}_f^2 - \frac{\omega}{2}\bar{\phi}_i^2 + iS_0(\bar{\phi}) + i\int_{t_i}^{t_f} dt J(t)\bar{\phi}(t)$$
$$-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi)$$
$$+(\dot{\bar{\phi}}(t_f) - \omega\bar{\phi}_f)\phi_f - (\dot{\bar{\phi}}(t_i) + \omega\bar{\phi}_i)\phi_i$$
$$+\int_{t_i}^{t_f} dt\phi[(-\partial_t^2 + \omega^2)\bar{\phi}(t) + J(t)] =$$

We choose $\bar{\phi}(t)$ in such a way that it eliminates the linear (black) term in the exponential
$\Rightarrow$ we get the differential equation

$$(\partial_t^2 - \omega^2)\bar{\phi}(t) = J(t)$$

with boundary conditions

$$\dot{\bar{\phi}}(t_f) = \omega\bar{\phi}_f, \qquad \dot{\bar{\phi}}(t_i) = -\omega\bar{\phi}_i$$

The solution of this equation is

$$\bar{\phi}(t) = \int_{t_i}^{t^f} G(t-t')J(t'), \qquad G_0(t-t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}$$

where $G_0(t-t')$ is the "propagator" for harmonic oscillator (see p. 11).

$$= -\int_{t_i}^{t^f} dtdt' J(t)G(t-t')J(t') - \frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi)$$

$\Rightarrow$ the numerator reduces to

$$\int d\phi_f d\phi_i \int D\phi e^{-\frac{\omega}{2}(\phi_f^2+\phi_i^2)} e^{iS_0+i\int_{t_i}^{t_f} dt J(t)\phi(t)} =$$

$$e^{-\frac{1}{2}\int_{t_i}^{t_f} dt dt' J(t) G(t-t') J(t')}$$
$$\int d\phi_f d\phi_i \int D\phi e^{-\frac{\omega}{2}(\phi_f^2+\phi_i^2)} e^{iS_0}$$

$$\Rightarrow \quad Z(J) = e^{-\frac{1}{2}\int_{t_i}^{t_f} dt dt' J(t) G(t-t') J(t')}$$

At $t_f \to \infty, t_i \to -\infty$ we get

$$Z(J) = e^{-\frac{1}{2}\int dt dt' J(t) G(t-t') J(t')}$$

Expanding this in powers of J we obtain

$$G_0(t_1,t_2) = G_0(t_1-t_2)$$
$$G_0(t_1,t_2,t_3,t_4) = G_0(t_1-t_2)G_0(t_3-t_4) +$$
$$G_0(t_2-t_3)G_0(t_1-t_4) + G_0(t_1-t_3)G_0(t_2-t_4)$$
$$G_0(t_1,t_2,t_3,t_4,t_5,t_6) =$$
$$G_0(t_1-t_2)G_0(t_3-t_4)G_0(t_5-t_6) + \ldots$$

while all the G_n with odd n vanish

$\Rightarrow$ Wick's theorem (see p. 12).

Functional integrals

Consider again lattice model for 1+1 Klein-Gordon field

$$\hat{H} = a\sum\Big[\frac{\hat{\pi}^2(x_k)}{2} + \frac{(\hat{\phi}(x_{k+1}) - \hat{\phi}(x_k))^2}{2a^2} + V(\phi_k)\Big]$$

where $V(\phi) = \frac{m^2}{2}$ for the free KG field of $V(\phi) = \frac{m^2}{2} + \lambda\phi^4$ for the self-interacting field. For one oscillator, we found the path integral representation for the evolution kernel

$$\langle\phi_f|e^{-i\hat{H}t}|\phi_i\rangle$$

where $|\phi\rangle$ were the eigenstates of the coordinate operator $\hat{\phi}$.

For 2N oscillators of our lattice model, the eigenstates of the coordinate operator $\hat{\phi}_n$ are

$$|\{\phi_K\}\rangle \equiv |\phi_{-N}\rangle|\phi_{-N-1}\rangle...|\phi_{-1}\rangle|\phi_0|\phi_1\rangle...|\phi_N\rangle$$

By construction, $|\{\phi_K\}\rangle$ are eigenstates of "field operator" $\widehat{\phi}_k$:

$$\widehat{\phi}_k|\{\phi_K\}\rangle = \phi_k|\{\phi_K\}\rangle$$

The evolution kernel is

$$\langle\{\phi_K\}^f|e^{-i\widehat{H}t_{fi}}|\{\phi_K\}^i\rangle$$

As in the case of harmonic oscillator, in order to find the path integral representation for the evolution kernel we divide t_{fi} into small intervals Δt and insert

$$1 = \int d\phi_{-N}|\phi_{-N}\rangle\langle\phi_{-N}|...\int d\phi_{-N}|\phi_{-N}\rangle\langle\phi_{-N}|$$
$$= \int \Pi d\phi_k|\{\phi_K\}\rangle\langle\{\phi_K\}|$$

We get

$$\langle\{\phi_K\}^f|e^{-i\widehat{H}t_{fi}}|\{\phi_K\}^i\rangle = \int \Pi d\phi_k^n \langle\{\phi_K\}^f|e^{-i\widehat{H}\Delta t}|\{\phi_K\}^n\rangle$$
$$\langle\{\phi_K\}^n|e^{-i\widehat{H}\Delta t}|\{\phi_K\}^{n-1}\rangle...\langle\{\phi_K\}^1|e^{-i\widehat{H}\Delta t}|\{\phi_K\}^i\rangle$$

For small Δt the evolution kernel for our lattice Hamiltonian is simply a product of evolution kernels for individual oscillators:

$$\langle\{\phi_K\}^{n+1}|e^{-i\hat{H}\Delta t}|\{\phi_K\}^n\rangle =$$
$$\left(\frac{1}{2\pi i\Delta t}\right)^N e^{ia\Delta t\sum_k\left[\frac{(\phi_k^{n+1}-\phi_k^n)^2}{2\Delta t^2}-\frac{(\phi_{k+1}^n-\phi_k^n)^2}{2a^2}-V(\phi_k^n)\right]}$$

$\Rightarrow$ functional integral for the KG field on the lattice

$$\langle\{\phi_K\}^f|e^{-i\hat{H}t_{fi}}|\{\phi_K\}^i\rangle = \left[\frac{1}{2\pi i\Delta t}\right]^{nN}$$
$$\int \Pi d\phi_k^n e^{ia\Delta t\sum_{k,n}\left[\frac{(\phi_k^{n+1}-\phi_k^n)^2}{2\Delta t^2}-\frac{(\phi_{k+1}^n-\phi_k^n)^2}{2a^2}-V(\phi_k^n)\right]}$$

As in the case of one oscillator, it is convenient to label the integration variables by the time t_n and position x_k rather than by n and k

$$\phi_k^n \to \phi(x_k, t_n) \qquad \Rightarrow$$

$$\langle \{\phi_K\}^f | e^{-i\hat{H}t_{fi}} | \{\phi_K\}^i \rangle = \left[\frac{1}{2\pi i \Delta t}\right]^{nN} \int \Pi d\phi(x_k, t_n)$$
$$\exp\left\{ ia\Delta t \sum_{k,n} \left[\frac{(\phi(x_k, t_n + \Delta t) - \phi(x_k, t_n))^2}{2\Delta t^2} \right.\right.$$
$$\left.\left. - \frac{(\phi(x_k + a, t_n) - \phi(x_k, t_n))^2}{2a^2} - V(\phi(x_k, t_n)) \right] \right\}$$

In the "continuum limit" $a, \Delta t \to 0$ we get

$$\langle \{\phi\}^f | e^{-i\hat{H}t_{fi}} | \{\phi\}^i \rangle =$$
$$\int D\phi(x,t) e^{i \int dxdt \left[\frac{(\dot{\phi}(x,t))^2}{2} - \frac{(\phi'(x,t))^2}{2a^2} - V(\phi(x,t)) \right]}$$

where $|\{\phi\}^i\rangle$ is a wave functional describing the state where the field is equal to $\phi(x,t)$.

The final form of the functional integral for the evolution kernel is

$$\langle \{\phi\}^f | e^{-i\hat{H}t_{fi}} | \{\phi\}^i \rangle = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi(x,t) e^{iS}$$

Because of the comlicated structure of the wave functional $|\{\phi\}^i\rangle$ it is more convenient to work in terms of Green functions where the initial and final states are simple (perturbative vacua).

Repeating the steps which lead us from the path integral for evolution kernel to path integrals for the Green functions, we get

$$\langle\Omega|T\{\hat{\phi}(x_1,t_1)...\hat{\phi}(x_n,t_n)|\Omega\rangle = \frac{\int D\phi(x,t)\phi(x_1,t_1)...\phi(x_n,t_n)e^{iS}}{\int D\phi(x,t)e^{iS}}$$

In four dimensions everything is the same (except we must start from 3-dimensional lattice) $\Rightarrow$

$$\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\int D\phi(x)\phi(x_1)...\phi(x_n)e^{iS}}{\int D\phi(x)e^{iS}}$$

The generating functional for Green functions

$$Z(J) = \frac{\int D\phi e^{iS_0 + i\int dx J(x)\phi(x)}}{\int D\phi e^{iS_0}}$$

As in the QM case, linear term in the numerator must vanish after the shift $\phi(x) \to \phi(x) + \bar{\phi}(x)$

$\Rightarrow$ the differential equation

$$(\partial^2 - m^2)\bar{\phi}(x) = J(x)$$

with the boundary conditions

$$\frac{\partial}{\partial t}\bar{\phi}(\vec{p},t) \overset{t\to\infty}{\Rightarrow} \omega_p \bar{\phi}(\vec{p},t)$$
$$\frac{\partial}{\partial t}\bar{\phi}(\vec{p},t) \overset{t\to\infty}{\Rightarrow} \omega_p \bar{\phi}(\vec{p},t),$$

reflecting the perurbative vacua at $t \to \pm\infty$.
$(\phi(\vec{p},t) \equiv \int d^3x e^{i\vec{x}\vec{p}}\phi(\vec{x},t))$
Solution:

$$\bar{\phi}(x) = \int dx' G_0(x-x')J(x')$$

where $G_0(x-x') = \int \frac{d^4p}{16\pi^4 i}\frac{1}{m^2-p^2-i\epsilon}e^{-ip(x-x')}$ is a free propagator $\Rightarrow$

$$Z(J) = e^{-\frac{1}{2}\int dx dx' J(x) G_0(x-x') J(x')}$$

Expanding this generating functional in powers of J one obtains Wick's theorem, just like for the anharmonic oscillator $\Rightarrow$

$$\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\int D\phi(x)\phi(x_1)...\phi(x_n)e^{iS}}{\int D\phi(x)e^{iS}}$$

= sum of Feynman diagrams

The Euclidean version of the functional integral for Green functions is

$$\langle\hat{\phi}(x_1)...\hat{\phi}(x_n)\rangle = \frac{\int D\phi(x)\phi(x_1)...\phi(x_n)e^{-S}}{\int D\phi(x)e^{-S}}$$

where the the boundary conditions are $\phi(\vec{x},t) \to 0$ at $t \to \pm\infty$. The correlation function $\langle\hat{\phi}(x_1)...\hat{\phi}(x_n)\rangle$ is the analytical continuation of the Green function $\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle$ to imaginary times $t_i \to -it_i$. The lattice form of the Euclidean functional integral is very convenient for computer calculations.

QED

Classical electrodynamics is a theory of electromagnetic field (describred by $F_{\mu\nu} = (\vec{E}, \vec{H})$ - field strength tensor) interacting with charged Dirac bispinor field $\psi(x)$.

First pair of Maxwell's eqs:

$$\partial_\mu F^{\mu\nu}(x) = e j^\nu(x)$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$
- 4-vector of the electomagnetic current (in particular, $ej_0 = e\psi^\dagger\psi \equiv \rho(x)$- charge density)

Second pair of Maxwell's eqns $\Leftarrow$ description in terms of potentials

$F_{\mu\nu} = \partial_\mu A_\nu - (\mu \leftrightarrow \nu)$

where $A_\mu = (\Phi, \vec{A})$ - scalar and vector potentials ("electromagnetic field").

The choice of potential is ambiguous $\Rightarrow$ gauge invariance: one can add $A_\mu \to A_\mu + \partial_\mu \alpha$ with an arbitrary $\alpha(x)$ and $\vec{E}$ and $\vec{H}$ will not notice it.

Dirac equation in an external electromagnetic field:

$$i\not{\mathcal{D}}\psi(x) = m\psi(x)$$

$\mathcal{D}_\mu = \partial_\mu + ieA_\mu$ - covariant derivative.
The electomagnetic coupling constant e is the charge of electron.

Electrodynamics Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(i\not{\mathcal{D}} - m)\psi(x)$$

Euler-Lagrange equation $\Rightarrow$ Maxwell's eqns + Dirac eqn.

Gauge invariance:

$$\left.\begin{array}{l} \psi(x) \to e^{i\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x) \\ A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \end{array}\right\} \Rightarrow \quad \mathcal{L}(x) \to \mathcal{L}(x)$$

Coulomb gauge:

$\partial_k A_k = 0, \qquad\qquad k = 1, 2, 3$

In Coulomb gauge Maxwell's eqns turn to

$$\vec{\partial}^2 \Phi(\vec{x}, t) = \rho((\vec{x}, t)), \qquad \partial^2 A_k(x) = j_k(x)$$

$\Rightarrow$ $A_0 \equiv \Phi$ is *not* an independent dynamical variable:

$$\Phi(\vec{x}, t) = e \int d^3x' \frac{\rho(\vec{x}', t)}{4\pi|\vec{x} - \vec{x}'|}$$ – Coulomb potential

The electromagnetic coupling constant e ($\equiv$ charge of the electon) is small

$$\frac{e^2}{4\pi} = \frac{1}{137} \qquad \left(\frac{e^2}{4\pi\hbar c} = \frac{1}{137}\right)$$

$\Rightarrow$ we can use perturbative expansion

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D + \mathcal{L}_{\text{int}}$$

$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$ –free e.m. Lagrangian
$\mathcal{L}_D = \bar{\psi}(x)(i \not{\partial} - m)\psi(x)$ –free Dirac Lagrangian
$\mathcal{L}_{\text{int}} = e\bar{\psi}(x) \not{A} \psi(x)$ –interaction Lagrangian

Quantization of the free e.m. field

$$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

We will quantize the electromagnetic field in the Coulomb gauge.

(+) - Non-physical degrees of freedom are absent.

(-) - Intermediate steps are not Lorentz invariant.

Without sources $A_0 \equiv 0 \Rightarrow$
we try $A_k(x)$ as canonical coordinates

Canonical momenta:

$$\left.\begin{array}{l} \pi^0 = \partial\mathcal{L}\partial\dot{A}_0 \quad - \quad \text{does not exist} \\ \pi^k = \partial\mathcal{L}\partial\dot{A}_k = -\dot{A}_k - \partial A_0 \partial x^k = -\dot{A}_k = E^k \end{array}\right\} \Rightarrow$$

$$H_F = \int d^3x (\pi^k \dot{A}_k - \mathcal{L}) = \int d^3x \frac{1}{2}(\vec{E}^2 + \vec{H}^2)$$

(recall that $\vec{E}^2 + \vec{H}^2$ is the energy density of e.m. field).

Quantization: we promote $A_k(\vec{x},t)$ and $\pi_k(\vec{x},t)$ to operators $\hat{A}_k(\vec{x})$ and $\hat{\pi}_k(\vec{x})$ satisfying the CCR

$$[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{\pi}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \equiv [\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

A problem: last line contradicts to Maxwell's eqs.

We want to have Gauss law $\vec{\nabla}\cdot\hat{\vec{E}} = 0$ as in classical physics, but

$$[\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ik}\delta^3(\vec{x}-\vec{y}) \Rightarrow$$
$$[\nabla\cdot\hat{\vec{E}}(\vec{x}), \hat{A}_j(\vec{y})] = i\partial_k\delta^3(\vec{x}-\vec{y}) \neq 0$$

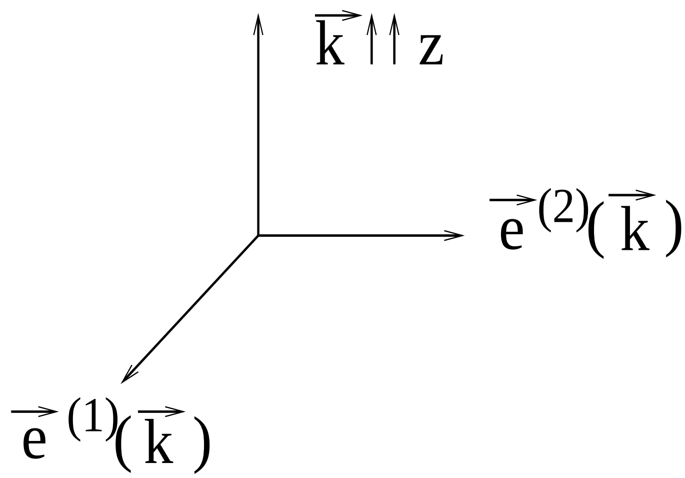
A trick that works in QED (but not in QCD):

$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta^{\mathrm{tr}}_{ik}\delta^3(\vec{x}-\vec{y})$$
$$\delta^{\mathrm{tr}}_{ik}\delta^3(\vec{x}-\vec{y}) \overset{\mathrm{def}}{\equiv} \int d^3k \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) e^{-i\vec{k}(\vec{x}-\vec{y})}$$

Expansion in ladder operators

$$\vec{\hat{A}}(\vec{x}) = \int \frac{d^3k}{2E_k} \sum_{\lambda=1,2} e^{\lambda}(\vec{k})\left(\hat{a}_k^{\lambda} e^{i\vec{k}\vec{x}} + \hat{a}_k^{\dagger\lambda} e^{-i\vec{k}\vec{x}}\right)$$

where $e^{\lambda}(\vec{k})$ - polarization vectors and $E_k = |\vec{k}|$



$$\left.\begin{array}{rcl} \left[\hat{a}_k^{\lambda}, \hat{a}_{k'}^{\lambda}\right] &=& 0 \\ \left[\hat{a}_k^{\dagger\lambda}, \hat{a}_{k'}^{\dagger\lambda}\right] &=& 0 \\ \left[\hat{a}_k^{\lambda}, \hat{a}_{k'}^{\dagger\lambda}\right] &=& (2\pi)^3\delta^3(\vec{k} - \vec{k}') \end{array}\right\} \Rightarrow \text{CCR}$$

$$\hat{H}_F = \frac{1}{2}\int d^3x(\vec{\hat{E}}^2 + \vec{\hat{H}}^2) = \int d^3k E_k \hat{a}_k^{\dagger\lambda}\hat{a}_k^{\dagger\lambda}$$

$$\text{CCR} \Rightarrow \left.\begin{array}{rcl} \left[\hat{H}_F, \hat{a}_k^{\dagger\lambda}\right] & = & E_k \hat{a}_k^{\dagger\lambda} \\ \left[\hat{H}_F, \hat{a}_k^{\lambda}\right] & = & -E_k \hat{a}_k^{\lambda} \end{array}\right\} \Rightarrow$$

$\Rightarrow \hat{a}^{\lambda}|0_F\rangle = 0$

$|0_F\rangle$ – the perturbative vacuum for free e.m. field (ground state of the Hamiltonian $\hat{H}_F$).

Heisenberg operators:

$$\vec{\hat{A}}(\vec{x}, t) = e^{i\hat{H}_F t} \vec{\hat{A}}(\vec{x}) e^{-i\hat{H}_F t}$$
$$\vec{\hat{\pi}}(\vec{x}, t) = e^{i\hat{H}_F t} \vec{\hat{\pi}}(\vec{x}) e^{-i\hat{H}_F t}$$

$$\Rightarrow \left.\begin{array}{r} {}^{i\hat{H}_F t} \hat{a}_k^{\lambda} e^{-i\hat{H}_F t} = \hat{a}_k^{\lambda} e^{-iE_k t} \\ e^{i\hat{H}_F t} \hat{a}_k^{\dagger\lambda} e^{-i\hat{H}_F t} = \hat{a}_k^{\dagger\lambda} e^{iE_k t} \end{array}\right\} \Rightarrow$$

$$\vec{\hat{A}}(x) = \int \frac{d^3k}{2E_k} \sum_{\lambda=1,2} e^{\lambda}(\vec{k}) \left(\hat{a}_k^{\lambda} e^{-ikx} + \hat{a}_k^{\dagger\lambda} e^{-ikx} \right)$$

Propagator of the transverse photons

$$D_{ij}^{\text{tr}} \equiv \langle 0_F | T\{\hat{A}_i(x) \hat{A}_j(y)\} | 0_F \rangle =$$
$$\int \frac{d^4k}{16\pi^4 i} e^{-ik(x-y)} \frac{1}{-k^2 - i\epsilon} \left(\delta_{ik} - \frac{k_i k_j}{\vec{k}^2} \right)$$

Quantization of the free Dirac field

$$\mathcal{L}_D = \bar{\psi}(x)(i\not\partial - m)\psi(x)$$

Canonical coordinate: $\psi(\vec{x},t)$
$\Rightarrow$ canonical momentum $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger$

$$H = \int d^3x\bar{\psi}(-i\vec{\gamma}\vec{\nabla} + m)\psi$$

Quantization: $\psi(\vec{x},t) \to \hat{\psi}(\vec{x})$, $\pi(\vec{x},t) \to \hat{\pi}(\vec{x})$
CAR (canonical anticommutation relations):

$$\{\hat{\psi}(\vec{x}),\hat{\psi}(\vec{y})\} = \{\hat{\psi}^\dagger(\vec{x}),\hat{\psi}^\dagger(\vec{y})\} = 0$$
$$\{\hat{\psi}(\vec{x}),\hat{\psi}^\dagger(\vec{y})\} = \delta^3(\vec{x}-\vec{y})$$

Ladder operators

$$\hat{\psi}(\vec{x}) = \int\frac{d^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat{a}^s_{\vec{p}}u(\vec{p},s)e^{i\vec{p}\vec{x}} + \hat{b}^{\dagger s}_{\vec{p}}v(\vec{p},s)e^{-i\vec{p}\vec{x}}\right]$$
$$\hat{\bar{\psi}}(\vec{x}) = \int\frac{d^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat{b}^s_{\vec{p}}\bar{v}(\vec{p},s)e^{i\vec{p}\vec{x}} + \hat{a}^{\dagger s}_{\vec{p}}\bar{u}(\vec{p},s)e^{-i\vec{p}\vec{x}}\right]$$

$u(\vec{p},s)$ – Dirac bispinor for the electron,
$v(\vec{p},s)$ – for the positron.

CAR for the ladder operators

$$\left.\begin{array}{ll}\{\hat{a}^s_{\vec{p}}, \hat{a}^{s'}_{\vec{p}}\} = 0 & \{\hat{b}^s_{\vec{p}}, \hat{b}^{s'}_{\vec{p}}\} = 0 \\ \{\hat{a}^{\dagger s}_{\vec{p}}, \hat{a}^{\dagger s'}_{\vec{p}'}\} = 0 & \{\hat{b}^{\dagger s}_{\vec{p}}, \hat{b}^{\dagger s'}_{\vec{p}'}\} = 0 \\ \{\hat{a}^s_{\vec{p}}, \hat{a}^{\dagger s'}_{\vec{p}'}\} = \delta^3(\vec{p} - \vec{p}')\delta_{ss'} & \{\hat{b}^s_{\vec{p}}, \hat{b}^{\dagger s'}_{\vec{p}'}\} = \delta^3(\vec{p} - \vec{p}')\delta_{ss'}\end{array}\right\} \Rightarrow$$

$$\{\hat{\psi}, \hat{\psi}\} = \{\hat{\bar{\psi}}, \hat{\bar{\psi}}\} = 0, \quad \{\hat{\psi}(\vec{x}), \hat{\bar{\psi}}(\vec{y})\} = \delta^3(\vec{x} - \vec{y})$$

$$\hat{H}_D = \int d^3x \hat{\bar{\psi}}(\vec{x})(-i\vec{\gamma}\vec{\nabla} + m)\bar{\psi}(\vec{x})$$

$$= \int d^3p E_p \sum_s \left(\hat{a}^{\dagger s}_{\vec{p}} \hat{a}^s_{\vec{p}} + \hat{b}^{\dagger s}_{\vec{p}} \hat{b}^s_{\vec{p}}\right)$$

CAR $\leftrightarrow$ CCR would lead to $+ \leftrightarrow -$

$\Rightarrow$ Hamiltonian would not have the ground state.

$$\left.\begin{array}{ll}\left[\hat{H}, \hat{a}^s_{\vec{p}}\right] = -E_p \hat{a}^s_{\vec{p}} & \left[\hat{H}, \hat{b}^s_{\vec{p}}\right] = -E_p \hat{b}^s_{\vec{p}} \\ \left[\hat{H}, \hat{a}^{\dagger s}_{\vec{p}}\right] = E_p \hat{a}^{\dagger s}_{\vec{p}} & \left[\hat{H}, \hat{b}^{\dagger s}_{\vec{p}}\right] = E_p \hat{b}^{\dagger s}_{\vec{p}}\end{array}\right\} \Rightarrow$$

$$\hat{a}^s_{\vec{p}} |0_D\rangle = \hat{b}^s_{\vec{p}} |0_D\rangle = 0$$

$|0_D\rangle \equiv$ the perturbative vacuum for Dirac field.

Heisenberg operators are defined as usual

$$\hat\psi(\vec x,t) = e^{i\hat H_D t}\hat\psi(\vec x)e^{-i\hat H_D t}, \quad \hat{\bar\psi}(\vec x,t) = e^{i\hat H_D t}\hat{\bar\psi}(\vec x)e^{-i\hat H_D t}$$

$$\left.\begin{array}{c} e^{i\hat H_D t}\hat a^s_{\vec p}e^{-i\hat H_D t} = \hat a^s_{\vec p}e^{-iE_p t} \\ e^{i\hat H_D t}\hat a^{\dagger s}_{\vec p}e^{-i\hat H_D t} = \hat a^{\dagger s}_{\vec p}e^{iE_p t} \\ \text{same for } \hat b \text{ and } \hat b^\dagger \end{array}\right\} \Rightarrow$$

$$\hat\psi(x) = \int \frac{d^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat a^s_{\vec p}u(\vec p,s)e^{-ipx} + \hat b^{\dagger s}_{\vec p}v(\vec p,s)e^{ipx}\right]$$

$$\hat{\bar\psi}(x) = \int \frac{d^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat b^s_{\vec p}\bar v(\vec p,s)e^{-ipx} + \hat a^{\dagger s}_{\vec p}\bar u(\vec p,s)e^{ipx}\right]$$

Propagator of the Dirac particle

$$S(x-y) = \langle 0_D|T\{\psi(x)\bar\psi(y)\}|0_D\rangle$$

$$T\{\psi(\hat x)\hat{\bar\psi}(y)\} \equiv \theta(x_0-y_0)\psi(\hat x)\hat{\bar\psi}(y) - \theta(y_0-x_0)\hat{\bar\psi}(y)\psi$$

CAR +

$\sum_s u(\vec p,s)\bar u(\vec p,s) = \not p + m, \ \sum_s v(\vec p,s)\bar v(\vec p,s) = \not p - m$

$$\Rightarrow \qquad S(x-y) = \int \frac{d^4p}{(2\pi)^4 i}\frac{m + \not p}{m^2 - p^2 - i\epsilon}e^{-ip(x-y)}$$

Quantization of electrodynamics in the Coulomb gauge

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \bar{\psi}(x)(i \not{D} - m)\psi(x)$$

Canonical coordinates: A_i, ψ, canonical momenta: $\pi^k = E^k, \pi = \psi^\dagger$.

$$H = H_D + H_F + H_{\text{int}} + H_{\text{Coulomb}}$$

$$\begin{aligned} H_{\text{int}}(t) &= e \int d^3x A^k(\vec{x},t) \bar{\psi}(\vec{x},t) \gamma_k \psi(\vec{x},t) \\ H_{\text{Coulomb}}(t) &= -\frac{e^2}{2} \int d^3x \Phi(\vec{x},t) \nabla^2 \Phi(\vec{x},t) \\ &= \int d^3x d^3y \rho(\vec{x},t) \frac{1}{4\pi|\vec{x}-\vec{y}|} \rho(\vec{y},t) \end{aligned}$$

$\rho(x) \equiv e\psi^\dagger(x)\psi(x)$
(Recall that $\Phi(\vec{x},t) = e \int d^3x' \frac{\rho(\vec{x}',t)}{4\pi|\vec{x}-\vec{x}'|}$)

Quantization: $A_i(\vec{x},t) \to \widehat{A}_i(\vec{x})$, $\psi(\vec{x},t) \to \widehat{\psi}(\vec{x})$, $\pi_i(\vec{x},t) \to \widehat{\pi}_i(\vec{x})$, $\pi(\vec{x},t) \to \widehat{\pi}(\vec{x})$.

Canonical (anti) commutaton relations – same as in a free theory.

QED Hamiltonian

$$\widehat{H} = \widehat{H}_D + \widehat{H}_{\text{int}} + \widehat{H}_{\text{Coulomb}}$$

$$\widehat{H}_{\text{int}} = e \int d^3x \widehat{A}_k(\vec{x}) \widehat{\bar{\psi}}(\vec{x}) \gamma_k \widehat{\psi}(\vec{x})$$

$$\widehat{H}_{\text{Coulomb}} = -\frac{e^2}{2} \int d^3x d^3y \widehat{\rho}(\vec{x}) \frac{1}{4\pi|\vec{x}-\vec{y}|} \widehat{\rho}(\vec{y})$$

$\widehat{H}_D$ leads to the Feynman propagator $S(x-y)$.
$\widehat{H}_F$ leads to the propagator of the transverse photons D^{tr}_{ij}

$$D^{\text{tr}}_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4 i} \frac{e^{-ik(x-y)}}{-k^2 - i\epsilon}$$
$$\left(-g_{\mu\nu} - \frac{k_\mu k_\nu + k_0(k_\mu \eta_\nu + \mu \leftrightarrow \nu)}{\vec{k}^2} - k^2 \frac{\eta_\mu \eta_\nu}{\vec{k}^2} \right)$$

$\eta = (1,0,0,0)$

- First (red) term is a Feynman propagator for the photon.
- Second (black) term does not contribute to physical matrix elements due to Ward identity

  Ward identity: Multiplication of the amplitude of the emission of the photon with momentum k by k_μ vanishes provided all the electrons and positrons are on the mass shell($\equiv p_i^2 = m^2$)
- Third (blue) term $= i\eta_\mu \eta_\nu \frac{\delta(x_0 - y_0)}{4\pi|\vec{x}-\vec{y}|}$ is the instantaneous interaction which cancels the contribution coming from $\hat{H}_{\text{Coulomb}}$

$\Rightarrow$

One can omit $\hat{H}_{\text{Coulomb}}$ from the Hamiltonian and use the rel.-inv. Feynman propagator

$$D^{\mathsf{F}}_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4 i} e^{-ik(x-y)} \frac{g_{\mu\nu}}{-k^2 - i\epsilon}$$

$$\langle\Omega|T\{\hat{\psi}(x_1)...\hat{\bar{\psi}}(x_n)A_\mu(y_1)...A_\nu(y_1)\}|\Omega\rangle = \frac{\langle 0|T\{\hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)e^{i^4x\hat{\mathcal{L}}_I dx}\}|0\rangle}{\langle 0|T\{e^{i^4x\hat{\mathcal{L}}_I dx}\}|0\rangle}$$

where $|0\rangle = |0_F\rangle|0_D\rangle$.

Expanding in powers of $\hat{\mathcal{L}}_I = e\hat{A}^I_\mu\hat{\bar{\psi}}_I\gamma^\mu\hat{\psi}_I$ one obtains a set of the correlation functions of the type

$$\langle 0|T\{\hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)\}|0\rangle = \sum_{\text{contractions}} \hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)$$

where

$$\hat{\psi}_I(x)\hat{\bar{\psi}}_I(x') = S(x-x') \quad \text{and} \quad A^I_\mu(y)A^I_\nu(y') = D^F_{\mu\nu}(y-y')$$

$$\Rightarrow \langle\Omega|T\{\hat{\psi}(x_1)...\hat{\bar{\psi}}(x_n)A_\mu(y_1)...A_\nu(y_1)\}|\Omega\rangle =$$

sum of Feynman diagrams with the photon propagator $D^F_{\mu\nu}(x-y)$, Dirac propagator $S(x-y)$, and the vertex γ_μ.

Fourier transformation $\Rightarrow$

Set of Feynman rules for QED Green functions in the momentum representation:

- $\frac{m+\not p}{m^2-p^2-i\epsilon}$ for each Dirac propagator
- $\frac{g_{\mu\nu}}{k^2+i\epsilon}$ for each photon propagator
- e for each electron-electron-photon vertex
- $\int \frac{d^4p}{16\pi^4 i}$ for each loop momentum.

The matrix elements of the S-matrix are obtained using the LSZ theorem.
Example: Compton scattering

$$G_{\mu\nu}(p_2, k_2; p_1, k_1) =$$
$$\frac{e^2}{k_1^2 k_2^2} \frac{m + \not{p}_2}{(m^2 - p_2^2)}$$
$$\left(\gamma_\mu \frac{m + \not{p}_1 + \not{k}_1}{m^2 - (p_1 + k_1)^2} \gamma_\nu + \gamma_\nu \frac{m + \not{p}_1 - \not{k}_2}{m^2 - (p_1 - k_2)^2} \gamma_\mu\right)$$
$$\frac{m + \not{p}_1}{(m^2 - p_1^2)}$$

LSZ theorem:

$$T(p_1, k_1; \lambda_1, s_1 \to p_2, k_2; \lambda_1, s_2) =$$
$$\lim_{k_i^2 \to 0} \lim_{p_i^2 \to m^2} k_1^2 e_\nu^{\lambda_1} k_2^2 e_\mu^{\lambda_2}$$
$$\bar{u}(p_2, s_2)(m + \not{p}_2) G_{\mu\nu}(p_2, k_2; p_1, k_1)(m + \not{p}_1) u(p_1, s_1)$$
$$= e^2 \bar{u}(p_2, s_2) \left(\not{e}^{\lambda_2} \frac{m + \not{p}_1 + \not{k}_1}{m^2 - (p_1 + k_1)^2} \not{e}^{\lambda_1} \right.$$
$$\left. + \not{e}^{\lambda_1} \frac{m + \not{p}_1 - \not{k}_2}{m^2 - (p_1 - k_2)^2} \not{e}^{\lambda_2} \right) u(p_1, s_1)$$

Cross section is $\sim |T|^2$.

A try on CCR in QED

$$[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})] = [\hat{\pi}_i(\vec{x}), \hat{\pi}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \equiv [\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

A problem: we want to have Gauss law $\vec{\nabla}\cdot\vec{\hat{E}} = 0$ as in classical physics, but

$$[\vec{\nabla}\cdot\vec{\hat{E}}(\vec{x}), \hat{A}_j(\vec{y})] = i\partial_k\delta^3(\vec{x}-\vec{y}) \neq 0$$

The trick that works for both QED and QCD (and for other gauge theories as well):

We impose the Gauss law on physical states instead of imposing it on the operators.

$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \;=\; i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

but

$$\vec{\nabla}\cdot\vec{\hat{E}}|\Psi_{\text{physical}}\rangle = 0$$

This still appears to contadict to CCR since

$$\langle\Psi_{\text{phys}}|[\nabla\cdot\vec{\hat{E}}(\vec{x}), \hat{A}_j(\vec{y})]|\Psi_{\text{phys}}\rangle = i\partial_k\delta^3(\vec{x}-\vec{y}) \neq 0$$
$$\langle\Psi_{\text{phys}}|\nabla\cdot\vec{\hat{E}}(\vec{x})\hat{A}_j(\vec{y}) - \hat{A}_j(\vec{y})\nabla\cdot\vec{\hat{E}}(\vec{x})|\Psi_{\text{phys}}\rangle = 0$$

but actually there is no contradiction since the l.h.s is ill-defined (see the QM examle below).

Baby version of a gauge theory

Consider a mechanical model with the Lagrangian ($x_{12} \equiv x_1 - x_2$)

$$L(A(t), x_1(t), x_2(t)) = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} + A^2 + (\dot{x}_1 + \dot{x}_2)A - \frac{\omega}{2} x_{12}^2$$

The Euler-Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} \quad \Rightarrow \quad \ddot{x}_1 + \dot{A} = \omega x_{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_1} \quad \Rightarrow \quad \ddot{x}_2 + \dot{A} = -\omega x_{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{A}} = \frac{\partial L}{\partial A} \quad \Rightarrow \quad 0 = \dot{x}_1 + \dot{x}_2 + 2A$$

This Lagrangian is invariant under the following "gauge transformation"

$$\begin{aligned} x_1(t) &\to x_1(t) + \alpha(t) \\ x_2(t) &\to x_2(t) + \alpha(t) \\ A(t) &\to A(t) - \dot{\alpha}(t) \end{aligned}$$

We can use this freedom to get rid of the variable A by choosing $\alpha(t) = \int dt A(t)$.

$$\Rightarrow \begin{cases} \ddot{x}_1 = \omega x_{12}, & \ddot{x}_2 = -\omega x_{12} \\ \dot{x}_1 + \dot{x}_2 = 0 \end{cases}$$

First two equations describe two particles with $m = 1$ connected by a spring and the last one means that the sum of their momenta is 0.

This problem may be described by the Lagrangian

$$L(x_1(t), x_2(t)) = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} - \frac{\omega}{2} x_{12}^2$$

PLUS

the additional requirement that the total momentum of the two particles vanishes:

$p_1(t) + p_2(t) = 0$.

This is an example of the constrained canonical system.

At first, let us forget about the constraint $p_1(t)+p_2(t)=0$.

New canonical coordinates:

$X = (x_1 + x_2)/2$ - coordinate of the c.m.

$x = x_{12}$ - separation

$$L(X(t), x(t)) = \dot{X}^2 + \frac{\dot{x}^2}{4} - \frac{\omega}{2}x^2$$

New canonical momenta:

$P = 2\dot{X} = p_1 + p_2$, $p = \frac{\dot{x}}{2} = \frac{1}{2}(p_1 - p_2)$

$$\Rightarrow H = \frac{P^2}{2} + p^2 - \frac{\omega}{2}x^2$$

Quantization:

$$\widehat{H} = \frac{\widehat{P}^2}{2} + \widehat{p}^2 - \frac{\omega}{2}\widehat{x}^2$$

Solutions of the Schrodinger eqn are

$$\Psi(X, x) = e^{iPX}\psi_n(x)$$

$\psi_n(x)$ - wavefunction of the n-th level of harmonic oscillator (Hermit polynomial).

Q: How to generalize the classical constraint that the observable $P = p_1 + p_2$ vanishes to quantum mechanics?

Wrong A: Require that the operator corresponding to this observable $\hat{P} = \hat{p}_1 + \hat{p}_2$ vanishes - contradicts to CCR $[p_i, x_j] = -i\delta_{ij}$.

Right A: Require that we consider only the "physical" states $\Psi_{\text{phys}} = \sum a_n \psi_n(x)$ with total momentum $P = 0 \to F(\hat{P})\Psi_{\text{phys}} = 0 \Rightarrow$we will observe $P = 0$ in all experiments.

Apparent "contradiction"

$$\langle \Psi_{\text{phys}} | [\hat{P}, \hat{X}] | \Psi_{\text{phys}} \rangle \quad = \quad 0 \quad \text{or} \quad i?$$

In explicit form ($|\Psi_{\text{phys}}\rangle = \Psi_{\text{vac}}$ for simplicity).

$$\int dx_1 dx_2 e^{-\frac{\omega}{2}x_{12}^2} \left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, x_1 + x_2 \right] e^{-\frac{\omega}{2}x_{12}^2} =$$
$$\int dx_1 dx_2 e^{-\frac{\omega}{2}x_{12}^2} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) (x_1 + x_2) e^{-\frac{\omega}{2}x_{12}^2}$$

which is 0 or ∞ depending on your taste (the integral is divergent along the coordinate $X = x_1 + x_2$).

Path integral for the constrained system

$$\langle \Psi^f_{\text{phys}} | e^{-i\hat{H}t} | \Psi^i_{\text{phys}} \rangle \quad = \quad ?$$

If we know how to solve the constraint,

$$\langle \Psi^f_{\text{phys}} | e^{-i\hat{H}t} | \Psi^i_{\text{phys}} \rangle =$$

$$\int dx_f dx_i \Psi^f_{\text{phys}}(x_f) \Psi^i_{\text{phys}}(x_i) \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t) e^{-\int_{t_i}^{t_f} \left(\frac{\dot{x}^2}{4} - \frac{\omega^2}{2} x^2 \right)}$$

Suppose we cannot solve the constraint (like in QCD). Still, our goal is the path integral with intermediate integrations over the dynamical coordinates only (we want to aviod the integrals divergent along the non-dynamical variable X). A way to achieve this is to repeat the derivation of the path integral, inserting at each t_k the projector on physical states

$$1_{\text{phys}} = \sum |\Psi_{\text{phys}}\rangle \langle \Psi_{\text{phys}}|$$

instead of the total set of states.

$1_{\text{phys}} = \int dp_1 dp_2 \delta(p_1 + p_2)|p_1\rangle|p_2\rangle\langle p_2|\langle p_1|$
$\int dx_1 dx_2 \delta(x_1 + x_2 - a)|x_1\rangle|x_2\rangle\langle x_2|\langle x_1|$ where a is an arbitrary number. Check:

$$\langle y_1|\langle y_2|1_{\text{phys}}|\Psi_{\text{phys}}\rangle =$$
$$\int dp_1 dp_2 \delta(p_1 + p_2) e^{-ip_1 y_1 - ip_2 y_2} \int dx_1 dx_2$$
$$e^{ip_1 x_1 + ip_2 x_2}\delta(x_1 + x_2 - a))\Psi_{\text{phys}}(x_{12}) = \Psi_{\text{phys}}(y_{12})$$

$$1_{\text{phys}} = \int \tilde{d}p_i \tilde{d}x_i e^{-ip_1 x_1 - ip_2 x_2}|p_1\rangle|p_2\rangle\langle x_1|\langle x_2|$$

$$\tilde{d}p_i \equiv dp_1 dp_2 \delta(p_1 + p_2), \;\; \tilde{d}x_i \equiv dx_1 dx_2 \delta(x_1 + x_2 - a)$$

Insert 1_{phys} n times:

$$\langle \Psi^f_{\text{phys}}|e^{-i\hat{H}t}\Psi^1_{\text{phys}}\rangle =$$
$$\langle \Psi^f_{\text{phys}}|e^{-i\hat{H}\Delta t}1_{\text{phys}} e^{-i\hat{H}\Delta t}1_{\text{phys}} \ldots e^{-i\hat{H}\Delta t}|\Psi^i_{\text{phys}}\rangle$$

$$e^{-i\hat{H}\Delta t}\Psi^1_{\text{phys}} =$$
$$\int \tilde{d}x^k_i \tilde{d}p^k_i e^{-ip^k_1 x^k_1 - ip^k_2 x^k_2} e^{-iH^k \Delta t}|p_1\rangle|p_2\rangle\langle x_1|\langle x_2|$$

$$H^k \equiv \frac{p^k_1}{2} + \frac{p^k_2}{2} - \frac{\omega^2}{2}x^2_k$$

$$e^{-i\hat{H}\Delta t}\Psi^1_{\text{phys}}e^{-i\hat{H}\Delta t}\Psi^1_{\text{phys}} =$$

$$\int \tilde{d}x_i^{k+1}\tilde{d}p_i^{k+1}|p_1^{k+1}\rangle|p_2^{k+1}\rangle e^{-i(p_1^{k+1}x_1^{k+1}+p_2^{k+1}x_2^{k+1}+H^{k+1}\Delta t)}$$

$$\int \tilde{d}x_i^k\tilde{d}p_i^k e^{i(p_1^k(x_1^{k+1}-x_1^k)+p_2^k(x_2^{k+1}-x_2^k)-H^k\Delta t)}\langle x_1^k|\langle x_2^k|$$

In the end of the day

$$\langle\Psi^f_{\text{phys}}|e^{-i\hat{H}t_{fi}}|\Psi^i_{\text{phys}}\rangle =$$

$$\int \tilde{d}x_f\Psi^f_{\text{phys}}(x^f_{12})\tilde{d}x_i\Psi^f_{\text{phys}}(x^i_{12})$$

$$\Pi\tilde{d}x_i^k\tilde{d}p_i^k e^{-i\left(p_1^{k+1}(x_1^{k+1}-x_1^k)+p_2^{k+1}(x_2^{k+1}-x_2^k)+H^{k+1}\Delta t\right)}$$

In the continuum limit this gives

$\langle\Psi$ … $\int$ …

$duetothemorecomplicatedstructureofWardidentityinQCI$

The meaning of the $\delta(X_k - a_k)$ is to restrict the integral over non-dynamical variables X_k:

$$\int \Pi dX_k \delta(X_k - a_k) = 1 \to \int \Pi dX_k dx_k \delta(X_k - a_k)\Psi(x_k) =$$

If the explicit form of this variables is unknown, one can use the arbitrary functions $f_k(X, x)$ $(X = X_1, ...X_n,\ x = x_1, ...x_n)$ because

$$\int \Pi dX_k \delta(f_k(X, x)\, \det \left| \frac{df_i(X, x)}{dX_j} \right| = 1$$

provided the equation $f_k(X, x) = 0$ has no multiple roots.

$$\frac{df_i(X, x)}{dX_j} = \{P_i, f_k\} \quad - \quad \text{Poisson brackets}$$

Definition:

$$\{F_i, G_j\} = \sum_k \frac{\partial F_i}{\partial P_k}\frac{\partial G_j}{\partial X_k} + \frac{\partial F_i}{\partial p_k}\frac{\partial G_j}{\partial x_k} - (F \leftrightarrow G)$$

Poisson brackets are invariant with respect to change of canonical coordinates $\Rightarrow$

$$\frac{df_i(X, x)}{dX_j} = \{(p_1 + p_2)_i, f_k(x_1, x_2)\}$$

In the continuum limit $f_k(x_1, x_2) \to f(x_1(t), x_2(t))$ and $\{p_1 + p_2)_i, f_k(x_1, x_2)\} \Rightarrow$
functional Poisson bracket
$\{(p_1 + p_2)(t), f(x_1(t'), x_2(t')\}$ which is a variational derivative:

$$\{F(t), G(t')\} \equiv \int dt \frac{\delta F}{\delta p_1(t)} \frac{\delta G}{\delta x_1(t')} + \frac{\delta F}{\delta p_2(t)} \frac{\delta G}{\delta x_2(t')} - (F \leftrightarrow$$

The final form of the path integral for the constrained system:

$$\langle \Psi^f_{\text{phys}} | e^{-i\hat{H}t_{fi}} | \Psi^i_{\text{phys}} \rangle =$$
$$\int \tilde{d}x_f \Psi^f_{\text{phys}}(x^f_{12}) \tilde{d}x_i \Psi^f_{\text{phys}}(x^i_{12})$$
$$\int Dp_i(t) \delta(p_1(t) + p_2(t)) \int Dx_i(t)$$
$$\det\{p_1(t) + p_2(t), f(x_1(t), x_2(t)\} \delta(f(x_1(t), x_2(t))$$
$$\exp\{i\,(p_1(t)\dot{x}_1(t) + p_2(t)\dot{x}_2(t) - H(t))\}$$

$\delta(p_1 + p_2)$ - Gauss law
$\delta(f(x_1(t), x_2(t))$ - “choice of gauge”

QCD

QCD is a theory of interacting quark and gluon fields and

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr} G_{\mu\nu}G^{\mu\nu} + \sum_{\text{flavors}} \bar{\psi}(i \not{D} - m)\psi_q$$

$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu - g[A_\mu, A_\nu]$, $D_\mu = \partial_\mu - igA_\mu$

g - coupling constant.

Gauge invariance:

$$\left.\begin{array}{l} \psi(x) \to S(x)\psi(x) \\ \bar{\psi}(x) \to S^\dagger(x)\bar{\psi}(x) \\ A_\mu(x) \to A_\mu(x) - \frac{i}{g}S^\dagger(x)\partial_\mu S(x) \end{array}\right\} \Rightarrow \ \mathcal{L}(x) \to \mathcal{L}(x)$$

$S(x)$ -an arbitrary $SU(3)$ matrix ($S = e^{i\alpha_a t^a}$, t^a- Gell-Mann matrices).

Classical theory: non-linear equations

$$D^\mu G_{\mu\nu} = j_\nu, \qquad (i \not{D} - m)\psi(x) = 0$$

Quantum theory $\Leftrightarrow$ perturbation theory (pQCD) + lattice simulations.

Perturbation theory - like QED:

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_F = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$\mathcal{L}_D \sum \bar{\psi}(i \not{\partial} - m)\psi_q$$

$$\mathcal{L}_{\text{int}} = -g \text{Tr} F_{\mu\nu}[A_\mu, A_\nu] + \sum \bar{\psi}_q \not{A} \psi_q$$

Free Lagrangian $\equiv$ 8 issues of electrodynamics labeled by $a = 1 \div 8$

$\Rightarrow$

Feynman rules are the same, except now we have the self-interaction of gluons.

This is almost true - Ward identity in QCD is different $\rightarrow$ ghosts.

Functional integral for QCD

Coulomb gauge: $\partial_k A_k^a = 0$.
Gauss law:

$$C = \vec{\nabla}\vec{E} + g[\vec{A}, \vec{E}]$$

$\hat{C}$ is a generator of gauge transformations:

$$e^{\int d^3x S(\vec{x})\hat{C}(\vec{x})}\hat{\psi}(\vec{x})e^{-\int d^3x S(\vec{x})\hat{C}(\vec{x})} = S(\vec{x})\hat{\psi}(\vec{x})$$

(for $A(\vec{x})$ - similarly)
$\rightarrow$ $\hat{C}$ should annihilate physical states:

$$\hat{C}(\vec{x})|\Psi_{\text{phys}}\rangle = 0$$

The functional integral (in pure QCD $\equiv$ gluodynamics) looks like in QM:

$x_i(t)$	$\rightarrow$	$A_i^a(\vec{x}, t)$		
$p_i(t)$	$\rightarrow$	$E_i^a(\vec{x}, t)$		
$\delta(p_1 + p_2)(t)$	$\rightarrow$	$\delta(C^a\vec{x}, t))$	$-$	Gauss law
$\delta(f(x_1(t), x_2(t))$	$\rightarrow$	$\delta(\partial_k A^{ak}\vec{x}, t))$	$-$	choice of gauge

$Z =$

$$\int DE_i^a(x) DA_i^a(x) \delta(\partial_k A^{ak}(x)) \delta(C^a(x))$$

$$\det\{C^a(x), \partial_k A^{bk}(y)\} e^{-i\int d^4x \left(\vec{E}^a(x)\cdot\dot{\vec{A}}^i(x) + \frac{1}{2}(\vec{E}^{a2} + \vec{H}^{a2})\right)}$$

The Poisson bracket is

$$\{C^a(x), \partial_k A^{bk}(y)\} = M\delta^4(x-y)$$
$$M = \nabla^2 \delta^{ab} - g f^{abc} \partial_k A^c(x)$$

The chromoelectric field $E_i^a(x)$ is an independent integration variable – we do not have the condition $E_i^a(x) = G_{i0}^a(x)$ yet.

The constrained δ-function can be written as a (functional) phase integral

$$\Pi\delta(C^a(x)) = \int DA_0^a e^{i\int d^4x A_0^a C^a(x)}$$

$\Rightarrow$ the Gaussian integration over $E_i^a(x)$ can be performed by shift $E_i^a(x) \to E_i^a(x) + G_{i0} \Rightarrow$

$$Z = \int DA^a_\mu(x)\delta(\partial_k A^{ak}(x))\,\text{det}\{M\delta(x-y)\}\, e^{-\frac{i}{4}\int d^4x\left(G^a_{\mu\nu}(x)G^{a\mu\nu}(x)\right)}$$

This is a functional integral for pure QCD (gluodynamics) in th Coulomb gauge. In the theory of functional integrals over the fermionic variables (Grassman functional integrals) it is proved that

$$\text{det}\, M = \int D\bar{c}(x)Dc(x)e^{i\int d^4x\bar{c}^a(x)M^abc^b(x)}$$

$\Rightarrow$ ghosts.

In Lorentz gauge

$$\partial_\mu A^{a\mu}(x) = 0$$

$\Rightarrow \delta(\partial_k A^{ak}(x))$ must be replaced by $\delta(\partial_\mu A^{a\mu}(x))$ and the operator M by

$$M_L = \partial^2\delta^{ab} - gf^{abc}\partial_\mu A^{c\mu}(x)$$

Functional integral for pure QCD in the Lorentz gauge

Z=

$$\int DA^a_\mu(x)\delta(\partial_\mu A^{a\mu}(x))\,\mathrm{det}\{M_L\delta(x-y)\}$$
$$e^{-\frac{i}{4}\int d^4x\left(G^a_{\mu\nu}(x)G^{a\mu\nu}(x)\right)}$$

$$\mathrm{det}\,M_L = \int D\bar{c}(x)Dc(x)e^{i\int d^4x\bar{c}^a(x)M^abc^b(x)}$$

$\Rightarrow$

$$Z =$$
$$\int DA^a_\mu(x)\int D\bar{c}(x)Dc(x)\delta(\partial_\mu A^{a\mu}(x))$$
$$e^{i\int d^4x\left(-\frac{1}{4}G^a_{\mu\nu}(x)G^{a\mu\nu}(x)+\bar{c}^a(x)\partial^2c^a(x)-g\partial_\mu\bar{c}^a(x)gf^{abc}A^{c\mu}(x)c^b(x)\right)}$$

$\Rightarrow$ ghosts propagate like scalar particles and interact with gluons. Ghosts live in loops only. Physical meaning: ghosts cancel the contributions of non-physical gluons which remain in loops.