1 Introduction: BFKL pomeron in high-energy pQCD
   - Regge limit in QCD.
   - Perturbative QCD at high energies.
   - BFKL and collider physics

2 High-energy scattering and Wilson lines
   - High-energy scattering and Wilson lines.
   - Evolution equation for color dipoles.
   - Light-ray vs Wilson-line operator expansion.
   - Rescaling in the Regge limit.
   - Propagators in a shock-wave background.
   - Leading order: BK equation.
Heisenberg uncertainty principle: $\Delta x = \frac{\hbar}{p} = \frac{\hbar c}{E}$

LHC: $E=7 \rightarrow 14$ TeV $\Leftrightarrow$ distances $\sim 10^{-18}$ cm
(Planck scale is $10^{-33}$ cm - a long way to go!)

To separate a “new physics signal” from the “old” background one needs to understand the behavior of QCD cross sections at large energies
Strong interactions at asymptotic energies: Froissart bound

Regge limit: $E \gg$ everything else

\[
\text{Causality, Unitarity} \quad \Rightarrow \quad \sigma_{\text{tot}} \xrightarrow{E \to \infty} \ln^2 E
\]

Froissart, 1962

Long-standing problem - not explained in any quantum field theory (or string theory) in 50 years!

Experiment: $\sigma_{\text{tot}} \sim s^{0.08}$ ($s \equiv 4E_{\text{c.m.}}^2$). Numerically close to $\ln^2 E$.
The pQCD process - Deep inelastic scattering

DIS: $e p \rightarrow e + X$

Asymptotic freedom: $\alpha_s(Q^2) \rightarrow 0$ as $Q^2 \rightarrow \infty$
Optical theorem: $\sigma_{\text{tot}} = \sum_{X} A_{ep \to p+X}^{\dagger} A_{ep \to p+X} = \Im A_{\text{forward}}$
Parton model

\[ \sigma_{\text{tot}} \sim \int d^4x \, e^{iq \cdot x} \langle N | j_\mu(x) j_\nu(0) | N \rangle \]

Parton model (leading order of pQCD):

\[ \sigma_{\text{tot}} \sim \sum_q e^2 q D_q(x_B), \quad x_B = \frac{Q^2}{2p \cdot q}, \quad q^2 = -Q^2 \]

\( D_q(x) = \) probability to find the quark with fraction \( x \) of nucleon’s momentum
Deep inelastic scattering in QCD

\[ D_q(x_B) \rightarrow D_q(x_B, Q^2) - \text{“scaling violations”} \]

DGLAP evolution (LLA(\(Q^2\))

\[ Q \frac{d}{dQ} D_q(x_B, Q^2) = K_{\text{DGLAP}} D_q(x_B, Q^2) \]

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-77

\[ K_{\text{DGLAP}} = \alpha_s(Q) K_{\text{LO}} + \alpha_s^2(Q) K_{\text{NLO}} + \alpha_s^3(Q) K_{\text{NNLO}} \ldots \]

The DGLAP equation sums up

\[ \sum_n \left( \alpha_s \ln \frac{Q^2}{m_N^2} \right)^n \left[ a_n + b_n \alpha_s + c_n \alpha_s^2 s + \ldots \right] \]

One fit at low \(Q_0^2 \sim 1 \text{ GeV}^2\) describes all the experimental data on DIS!
Deep inelastic scattering at small $x_B$

Regge limit in DIS: $E \gg Q \equiv x_B \ll 1$

DGLAP evolution $\equiv Q^2$ evolution

$$Q \frac{d}{dQ} D_g(x_B, Q^2) = K_{\text{DGLAP}} D_g(x_B, Q^2)$$

Not really a theory - needs the $x$-dependence of the input at $Q_0^2 \sim 1\text{GeV}^2$
Deep inelastic scattering at small $x_B$

**Regge limit in DIS:** $E \gg Q \equiv x_B \ll 1$

**DGLAP evolution** $\equiv Q^2$ evolution

\[ Q \frac{d}{dQ} D_g(x_B, Q^2) = K_{DGLAP} D_g(x_B, Q^2) \]

Not really a theory - needs the $x$-dependence of the input at $Q_0^2 \sim 1\text{GeV}^2$

**BFKL evolution** $\equiv x_B$ evolution (Balitsky, Fadin, Kuraev, Lipatov, 1975-78)

\[ \frac{d}{dx_B} D_g(x_B, Q^2) = K_{BFKL} D_g(x_B, Q^2) \]

Theory, but with problems
In pQCD: Leading Log Approximation ⇒ BFKL pomeron

\[ s = (p_A + p_B)^2 \approx 4E^2 \]

Leading Log Approximation (LLA(x)):

\[ \alpha_s \ll 1, \quad \alpha_s \ln s \sim 1 \]
In pQCD: Leading Log Approximation $\Rightarrow$ BFKL pomeron

\[ s = (p_A + p_B)^2 \simeq 4E^2 \]

Leading Log Approximation (LLA($x$)):

\[ \alpha_s \ll 1, \quad \alpha_s \ln s \sim 1 \]

The sum of gluon ladder diagrams gives

\[ \sigma_{\text{tot}} \sim s^{12} \frac{\alpha_s}{\pi} \ln 2 \]

Numerically: for DIS at HERA

\[ \sigma \sim s^{0.3} = x_B^{-0.3} \]

- qualitatively OK
In pQCD: Leading Log Approximation $\Rightarrow$ BFKL pomeron

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Numerically: for DIS at HERA

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- qualitatively OK
BFKL vs HERA data

\[ F_2(x_B, Q^2) = c(Q^2)x_B^{-\lambda(Q^2)} \]

M.Hentschinski, A. Sabio Vera and C. Salas, 2010
Collinear factorization (LLA($Q^2$)):

$$\sigma_H = \int dx_1 dx_2 D_g(x_1, m_H) D_g(x_2, m_H) \sigma_{gg \rightarrow H}$$

sum of the logs \( (\alpha_s \ln \frac{m_X^2}{m_N^2})^n \), \( \ln \frac{s}{m_X^2} \sim 1 \)
Collinear factorization (LLA($Q^2$)):

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- sum of the logs \( (\alpha_s \ln \frac{m_X^2}{m_N^2})^n \), \( \ln \frac{s}{m_X^2} \sim 1 \)

LLA($x$): $k_T$-factorization

$$\sigma_H = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg \rightarrow H}$$

- sum of the logs \( (\alpha_s \ln x_i)^n \), \( \ln \frac{m_X^2}{m_N^2} \sim 1 \)

Much less understood theoretically.
Collinear factorization (LLA($Q^2$)):

\[ \sigma_H = \int dx_1 dx_2 D_g(x_1, m_H) D_g(x_2, m_H) \sigma_{gg \rightarrow H} \]

sum of the logs \( (\alpha_s \ln \frac{m_H^2}{m_N^2})^n, \quad \ln \frac{s}{m_X^2} \sim 1 \)

LLA(x): $k_T$-factorization

\[ \sigma_H = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg \rightarrow H} \]

- sum of the logs \( (\alpha_s \ln x_i)^n, \quad \ln \frac{m_X^2}{m_N^2} \sim 1 \)

Much less understood theoretically.

For Higgs production in the central rapidity region \( x_{1,2} \sim \frac{m_H}{\sqrt{s}} \sim 0.01 \) and we know from DIS experiments that at such \( x_B \) the DGLAP formalism works pretty well \Rightarrow no need for BFKL resummation
Collinear factorization (LLA($Q^2$)):

\[ \sigma_H = \int dx_1 dx_2 D_g(x_1, m_H)D_g(x_2, m_H)\sigma_{gg \rightarrow H} \]

sum of the logs \( (\alpha_s \ln \frac{m_X^2}{m_N^2})^n \), \( \ln \frac{s}{m_X^2} \sim 1 \)

LLA(x): $k_T$-factorization

\[ \sigma_H = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A)g(k_2^\perp, x_B)\sigma_{gg \rightarrow H} \]

- sum of the logs \( (\alpha_s \ln x_i)^n \), \( \ln \frac{m_X^2}{m_N^2} \sim 1 \)

Much less understood theoretically.

For \( m_X \sim 10\text{GeV} \) (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well \( \Rightarrow \) some kind of BFKL resummation is needed.
MHV gluon amplitudes ⇔ light-like Wilson-loop polygons

Alday, Maldacena (at large $\alpha_sN_c$)

Checked up to 6 gluons/2 loops (Korchemsky et. al).
Uses of BFKL: MHV amplitudes in $\mathcal{N} = 4$ SYM

MHV gluon amplitudes $\Leftrightarrow$ light-like Wilson-loop polygons

Alday, Maldacena (at large $\alpha_s N_c$)

\[ p_i = x_i - x_{i-1} \]
\[ p_i^2 = 0 \]

Checked up to 6 gluons/2 loops (Korchemsky et. al).

BDS ansatz: $\ln A_{\text{MHV}} = \text{IR terms} + F_n, \quad F_n = \Gamma_{\text{cusp}}(\text{angles}) + (F_n^1 + R_n)$

BFKL in multi-Regge region $\Rightarrow$ asymptotics of remainder function $R_n$ (Lipatov et al)
Structure functions of DIS are determined by matrix elements of twist-2 operators

\[ \mathcal{O}_G^{(j)} = F_{\mu_1\xi} D_{\mu_2} \ldots D_{\mu_{j-1}} F_{\mu_j} \]

\[ \mu^2 \frac{d}{d\mu^2} \mathcal{O}_G^{(j)} = \frac{\gamma(j)(\alpha_s)}{4\pi} \mathcal{O}_G^{(j)} \]

BFKL gives asymptotics of $\gamma(j)$ at $j \to 1$ in all orders in $\alpha_s$

\[ \gamma(j) = \sum_n \left( \frac{\alpha_s}{j-1} \right)^n \left[ C^{(n)}_{\text{LO BFKL}} + \alpha_s C^{(n)}_{\text{NLO BFKL}} \right] \]

Checked by explicit calculation of Feynman diagrams up to 3 loops in QCD and $\mathcal{N} = 4$ SYM. (Janik et al)

Integrability of spin chains corresponding to evolution of $\mathcal{N} = 4$ SYM operators $\Rightarrow \gamma(j)$ in 5 loops agrees with BFKL (Janik et al).

For all order of pert. theory: Y-system of equations (Gromov, Kazakov, Viera). Hopefully agrees with BFKL.
Towards the high-energy QCD

\[ \sigma_{\text{tot}} \sim s^{12} \frac{\alpha_s}{\pi} \ln 2 \] violates Froissart bound
\[ \sigma_{\text{tot}} \leq \ln^2 s \Rightarrow \text{pre-asymptotic behavior.} \]

True asymptotics as \( E \to \infty = ? \)

Possible approaches:

- Sum all logs \( \alpha_s^m \ln^n s \)
- Reduce high-energy QCD to 2 + 1 effective theory
Towards the high-energy QCD

$$\sigma_{\text{tot}} \sim s^{12} \frac{\alpha_s}{\pi} \ln 2$$ violates Froissart bound $\sigma_{\text{tot}} \leq \ln^2 s$\n
$\Rightarrow$ pre-asymptotic behavior.

True asymptotics as $E \to \infty = ?$

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to $2 + 1$ effective theory

Lecture II: NLO corrections $\alpha_s^{n+1} \ln^n s$
WKB approximation: $\Psi \sim e^{iS/\hbar}$

$$S = \int (pdz - Edt)$$

$$= -Et + \int^{z} dz' \sqrt{2m(E - V(z'))}$$
WKB approximation: \( \Psi \sim e^{\frac{i}{\hbar}S} \)

\[
S = \int (p\,dz - E\,dt)
\]
\[
= -Et + \int_{z'}^{z} \sqrt{2m(E - V(z'))} \, dz'
\]

High energy: \( E \gg V(x) \Rightarrow \)

\[
\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et-kx)} e^{-\frac{i}{\hbar} \int_{-\infty}^{z} dz' V(z')}
\]
High-energy scattering and “Wilson lines” in quantum mechanics

WKB approximation: \( \Psi \sim e^{\frac{i}{\hbar}S} \)

\[
S = \int (pdz - Edt) = -Et + \int_{z'}^{z} dz' \sqrt{2m(E - V(z'))}
\]

High energy: \( E \gg V(x) \Rightarrow \)

\[
\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et-kx)} e^{-\frac{i}{\hbar} \int_{-\infty}^{z'} dz' V(z')}
\]

\( \Psi \) at high energy = free wave \( \times \) phase factor ordered along the line \( \parallel \vec{v} \).
High-energy scattering and “Wilson lines” in quantum mechanics

WKB approximation: \( \Psi \sim e^{i \frac{i}{\hbar} S} \)

\[
S = \int (pdz -.Edt)
= -Et + \int^{z} dz' \sqrt{2m(E - V(z'))}
\]

High energy: \( E \gg V(x) \Rightarrow \)

\[
\Psi(\vec{r}, t) = e^{-i \frac{Et - kx}{\hbar}} e^{-i \frac{i}{\hbar} \int_{-\infty}^{z} dz' V(z')}
\]

\( \Psi \) at high energy = free wave \( \times \) phase factor ordered along the line \( || \vec{v} \).

The scattering amplitude is proportional to \( \Psi(t = \infty) \) defined by

\[
U(x_{\perp}) = e^{-i \frac{i}{\hbar} \int_{-\infty}^{\infty} dz' V(z' + x_{\perp})}
\]

Glauber formula: \( \sigma_{\text{tot}} = 2 \int d^{2}x_{\perp} \left[ 1 - \Re U(x_{\perp}) \right] \)
High-energy phase factor in QED and QCD

classical trajectory: \( \vec{r} = \vec{v}t \)

\[
S_e = \int dt \left\{ -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A} \right\} = S_{\text{free}} + \int dt (-e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A})
\]

\( \Rightarrow \) phase factor for the high-energy scattering is

\[
U(x_\perp) = e^{-i\frac{e}{\hbar c} \int_{-\infty}^{\infty} dt (-e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A})} = e^{-i\frac{e}{\hbar c} \int_{-\infty}^{\infty} dt \, \dot{x}_\mu A^\mu (x(t))}
\]
High-energy phase factor in QED and QCD

classical trajectory: $\vec{r} = \vec{v}t$

$S_e = \int dt \left\{ -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v} \cdot \vec{A} \right\} = S_{\text{free}} + \int dt (-e\Phi + \frac{e}{c}\vec{v} \cdot \vec{A})$

$\Rightarrow$ phase factor for the high-energy scattering is

$$U(x_\perp) = e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty} dt (-e\Phi + \frac{e}{c}\vec{v} \cdot \vec{A})} = e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty} dt \dot{x}_\mu A^\mu(x(t))}$$

In QCD $e \rightarrow -g$, $A_\mu \rightarrow A_\mu \equiv A_\mu^a t^a$

$t^a$ - color matrices

$\Rightarrow U(x_\perp, v) = P\exp\left\{ \frac{ig}{\hbar c} \int_{-\infty}^{\infty} dt \dot{x}_\mu A^\mu(x(t)) \right\}$

(Later $\hbar = c = 1$)

Wilson − line operator
At high energies, particles move along straight lines \( \Rightarrow \) the amplitude of \( \gamma^* A \rightarrow \gamma^* A \) scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):

\[
\mathcal{A}(s) = \int d^2 k_\perp 4\pi^2 I_A(k_\perp) \langle B | \text{Tr} \left\{ U(k_\perp) U^\dagger(-k_\perp) \right\} | B \rangle
\]

Formally, means the operator expansion in Wilson lines.
At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^*A \rightarrow \gamma^*A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):

\[
A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr} \{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle
\]

Formally, $\Rightarrow$ means the operator expansion in Wilson lines.
Light-cone expansion and DGLAP evolution in the NLO

\[ k^2_\perp > \mu^2 \]

\[ k^2_\perp < \mu^2 \]

\( \mu^2 \) - factorization scale (normalization point)

\( k^2_\perp > \mu^2 \) - coefficient functions

\( k^2_\perp < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))
\( \kappa^2 > \mu^2 \) - coefficient functions

\( \kappa^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

**OPE in light-ray operators**

\[
T \{ j_\mu(x) j_\nu(y) \} = \frac{x^2}{2\pi^2 x^4} \left[ 1 + \frac{\alpha_s}{\pi} (\ln x^2 \mu^2 + C) \right] \bar{\psi}(x) \gamma_\mu \gamma_\xi \gamma_\nu [x, y] \psi(y) + O(\frac{1}{x^2})
\]

\([x, y] = Pe^{ig \int_0^1 du (x-y) \mu A_\mu (ux+(1-u)y)} \) - gauge link
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \]

\[ k_\perp^2 < \mu^2 \]

\( \mu^2 \) - factorization scale (normalization point)

\( k_\perp^2 > \mu^2 \) - coefficient functions

\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\[ (x - y)^2 = 0 \]

\[ \mu^2 \frac{d}{d\mu^2} \bar\psi(x)[x, y]\psi(y) = K_{\text{LO}} \bar\psi(x)[x, y]\psi(y) + \alpha_s K_{\text{NLO}} \bar\psi(x)[x, y]\psi(y) \]
Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude.
At high energies, particles move along straight lines \( \Rightarrow \)
the amplitude of \( \gamma^* A \rightarrow \gamma^* A \) scattering reduces to the matrix element of
a two-Wilson-line operator (color dipole):

\[
A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr} \{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle
\]

\[
U(x_\perp) = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du \ n^\mu A_\mu(un + x_\perp) \right]
\]
At high energies, particles move along straight lines ⇒
the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr} \{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle$$

$$U(x_\perp) = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du \ n^\mu A_\mu(un + x_\perp) \right]$$

Formally, $\rightarrow$ means the operator expansion in Wilson lines
Rapidity factorization

η - rapidity factorization scale

Rapidity $Y > \eta$ - coefficient function ("impact factor")
Rapidity $Y < \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by $\eta$

$$U_x^\eta = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$
Wilson lines from Feynman diagrams

\[ p = \alpha p_1 + \beta p_2 + p_\perp \]
- Sudakov variables.

\[ g_{\mu\nu} \rightarrow \frac{2}{s} p_{1\mu} p_{2\nu} \]

I will prove now that if I replace this by the “eikonal propagator”

\[ \frac{\not{k} - \not{p}}{(k - p)^2 + i\epsilon} \]

\[ \simeq \frac{\not{p}_2}{\beta_k - \beta_p - \frac{(\vec{k} - \vec{p})^2_\perp}{\alpha_k s} + i\epsilon\alpha_k} . \]

I will prove now that if I replace this by the “eikonal propagator”

\[ \frac{\not{p}_2}{-\beta_p + i\epsilon\alpha_k} , \]

the value of the loop integral over \( \beta_p \) remains unchanged.
Wilson lines from Feynman diagrams

\[ k \quad k-p \quad k-p' \]

\[ l \]

\[ p+\| \quad p+\|p' \]

\[ p = \alpha p_1 + \beta p_2 + p_\perp \]

- Sudakov variables.

\[ g_{\mu\nu} \rightarrow \frac{2}{s} p_{1\mu} p_{2\nu} \]

1. Residue at the pole of quark propagator

\[ \beta_p = \beta_k - \frac{(\vec{k} - \vec{p})^2_\perp}{\alpha_k s} \]

Gluon:

\[ (\alpha_p + \alpha_l) \beta_{l\perp} - (p + \vec{p})^2_\perp + (\alpha_p + \alpha_l) \beta_{k\perp} - \frac{\alpha_p + \alpha_l}{\alpha_k} (\vec{k} - \vec{p})^2_\perp. \]

First two terms \( \sim m^2 \) while the second two \( \sim \frac{\alpha_p}{\alpha_k} m^2 \) \( (\beta_k \sim \frac{m^2}{\alpha_k s}) \)

\( \Rightarrow \) same result as from the pole at \( \beta_p = 0 \).
Wilson lines from Feynman diagrams

\[ M(k-p, k-p') = k \cdot p \cdot k-p \cdot k-p' \]

2. Residue at the pole of a gluon propagator $\beta_p = -\beta_l + \frac{(p+l)^2}{(\alpha_p + \alpha_l)s}$

Quark prop:

\[
\beta_l - \frac{(p+l)^2}{(\alpha_p + \alpha_l)s} + \beta_k - \frac{(k-p)^2}{\alpha_k s} + i\epsilon \alpha_k \quad \rightarrow \quad \frac{\psi_2}{\beta_l - \frac{(p+l)^2}{(\alpha_p + \alpha_l)s}}
\]

(first two terms $\sim \frac{m^2}{\alpha_p s}$, second two terms $\sim \frac{m^2}{\alpha_k s}$) $\Leftrightarrow$ quark pole $\frac{p^\mu}{-\beta_p + i\epsilon \alpha_k}$

$\alpha_k \gg \alpha_p, \alpha_l$

$p = \alpha p_1 + \beta p_2 + p_\perp$

- Sudakov variables.

\[ g_{\mu\nu} \rightarrow \frac{2}{s} p_1\mu p_2\nu \]
Rapidity factorization

η - rapidity factorization scale

Rapidity $Y > \eta$ - coefficient function ("impact factor")
Rapidity $Y < \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by $\eta$

\[ U_x^\eta = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right] \]

\[ A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik\cdot x} A_\mu(k) \]
Each path is weighted with the gauge factor $P e^{i g \int d \mu A_{\mu}}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$[ x \rightarrow z$: free propagation$] \times$

$[U^{ab}(z_{\perp})$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$] \times$

$[ z \rightarrow y$: free propagation$]$
Rescaling in the Regge limit

Amplitude = correlation function of 4 scalar currents

\[ A(s, t) = -i \int d^2 z_\perp e^{-i(r,z)_\perp} N^{-1} \int DA \, e^{iS(A)} \det(i\nabla) \times \left\{ \int dz_+ \int d^4 x \, e^{-ip_A \cdot x} \langle j(x_-, x_+ + z_+, x_\perp + z_\perp) j(0, z_+, z_\perp) \rangle_A \right\} \times \left\{ \int dz_- \int d^4 y \, e^{-ip_B \cdot y} \langle j(y_- + z_-, y_+, y_\perp) j(z_-, 0, 0_\perp) \rangle_A \right\} , \]

Regge limit: \( s = (p_A + p_B)^2 \to \infty, p_A^2, p_B^2, t = -r_\perp^2 \) - fixed

\[ p_A = \lambda \kappa e_- + \frac{p_A^2}{s} \kappa e_+, \quad e_+ \cdot e_- = 1, \quad \kappa \equiv \sqrt{\frac{s}{2}} \]

\[ p_A = \lambda \kappa e_+ + \frac{p_B^2}{s} \kappa e_- , \]
**“External”** shock-wave gluon field

We “freeze” the gluon field, consider the “upper part”

\[
\int dz_+ \int d^4 x \ e^{-ip_A \cdot x} \langle j(x^-, x^+ + z^+, x_\perp + z_\perp) j(0, z^+, z_\perp) \rangle_A
\]

and rescale \( z_+ \rightarrow \lambda z_+ \), \( z_- \rightarrow \frac{z_-}{\lambda} \)

\[
(p_A^{(0)} = \kappa e_+ + \frac{p_A^2}{s} \kappa e_- )
\]

\[
\int d^4 x \ d^4 z \ \delta(z_-) e^{-ip_A x - i(r,z)_\perp} \langle j(x + z) j(z) \rangle_A
\]

\[
= \lambda \int d^4 x \ d^4 z \ \delta(z_-) e^{-ip_A^{(0)} x - i(r,z)_\perp} \langle j(x + z) j_\nu(z) \rangle_B,
\]

The boosted field \( B_\mu \) has the form

\[
B_-(x^-, x^+, x_\perp) = \lambda A_-(\frac{x^-}{\lambda}, \lambda x^+, x_\perp),
\]

\[
B_*(x^-, x^+, x_\perp) = \frac{1}{\lambda} A_+(\frac{x^-}{\lambda}, \lambda x^+, x_\perp),
\]

\[
B_\perp(x^-, x^+, x_\perp) = A_\perp(\frac{x^-}{\lambda}, \lambda x^+, x_\perp),
\]
"External" shock-wave gluon field

We “freeze” the gluon field, consider the “upper part"

$$\int dz_+ \int d^4x e^{-ipA \cdot x} \langle j(x_-, x_+ + z_+, x_\perp + z_\perp) j(0, z_+, z_\perp) \rangle_A$$

and rescale $z_+ \rightarrow \lambda z_+$, $z_- \rightarrow \frac{z_-}{\lambda}$

$$\int d^4x \int d^4z \delta(z_-) e^{-ipAx - i(r,z)_\perp} \langle j(x + z)j(z) \rangle_A$$

$$= \lambda \int d^4x \int d^4z \delta(z_-) e^{-ipA^{(0)}x - i(r,z)_\perp} \langle j(x + z)j_\nu(z) \rangle_B,$$

If $F_{\mu\nu}(A) \rightarrow 0$ as $x_+ \rightarrow \infty$ we get a “pancake” field for $G_{\mu\nu}(B)$

$$G_{-i}(x_-, x_+, x_\perp) = \lambda F_{-i}(\frac{x_-}{\lambda}, \lambda x_+, x_\perp) \rightarrow \delta(x_+) G_i(x_\perp),$$

$$G_{+i}(x_-, x_+, x_\perp) = \frac{1}{\lambda} F_{+i}(\frac{x_-}{\lambda}, \lambda x_+, x_\perp) \rightarrow 0,$$

$$G_{+-,ik}(x_-, x_+, x_\perp) = \frac{1}{\lambda} F_{+-,ik}(\frac{x_-}{\lambda}, \lambda x_+, x_\perp) \rightarrow 0,$$
We “freeze” the gluon field, consider the “upper part”

\[
\int d\zeta_+ \int d^4x e^{-ip_A \cdot x} \langle j(x_-, x_+ + \zeta_+, x_\perp + \zeta_\perp) j(0, \zeta_+, \zeta_\perp) \rangle_A
\]

and rescale \( \zeta_+ \to \lambda \zeta_+ \), \( \zeta_- \to \frac{\zeta_-}{\lambda} \)

\[
\left( p_A^{(0)} = \kappa e_+ + \frac{p_A^2}{s} \kappa e_- \right)
\]

\[
\int d^4x d^4\zeta \delta(\zeta_-) e^{-ip_A x - i(r,\zeta)_\perp} \langle j(x + \zeta) j(\zeta) \rangle_A
\]

\[
= \lambda \int d^4x d^4\zeta \delta(\zeta_-) e^{-ip_A^{(0)} x - i(r,\zeta)_\perp} \langle j(x + \zeta) j_\nu(\zeta) \rangle_B,
\]

The only component which survives the infinite boost is \( F_{-\perp} \) and it exists only within the thin “pancake” near \( x_+ = 0 \). In the rest of the space the field \( B_{\mu} \) is a pure gauge. Let us denote by \( \Omega \) the corresponding gauge matrix and by \( B^{\Omega} \) the rotated gauge field which vanishes everywhere except the pancake:

\[
B^{\Omega}_- = \lim_{\lambda \to \infty} \frac{\partial^i}{\partial_\perp^2} G^{\Omega}_{i-}(0, \lambda x_*, x_\perp) \to \delta(x_*) \frac{\partial^i}{\partial_\perp^2} G^{\Omega}_i(x_\perp), \quad B^{\Omega}_+ = B^{\Omega}_\perp = 0.
\]
Propagators in the shock-wave background

“Pancake” is very thin
\( (l_+ \sim \frac{1}{\lambda}) \Rightarrow \) path inside the shock wave can be approximated by a segment of the straight line in \( x_+ \) direction

\[
(x\left| \frac{1}{P^2} \right| y) = \frac{-i}{\mathcal{N}} \int_0^\infty d\tau \int_{x(0)=y}^{x(\tau)\equiv x} D x(t) e^{-i\int_0^\tau dt \frac{x^2}{4}} \text{Pexp}\left\{ig \int_0^\tau dt (B^\Omega_\mu(x(t))\dot{x}^\mu(t)) \right\}
\]

\[
= \int \frac{d^4z}{4\pi^4} \delta(z_+) \frac{1}{(x-z)^2} \frac{\leftrightarrow}{\partial z_-} \frac{1}{(z-y)^2} \text{Pexp}\left\{ig \int dz_+ B^\Omega_-(z_+, z_-) \right\}
\]
“Pancake” is very thin \((l_+ \sim \frac{1}{\lambda})\) \(\Rightarrow\) path inside the shock wave can be approximated by a segment of the straight line in \(x_+\) direction

\[
(x\bigg\vert \frac{1}{P^2}\bigg\vert y) = \frac{-i}{\mathcal{N}} \int_0^\infty d\tau \int_{x(0)=y}^{x(\tau)=x} \mathcal{D}x(t) e^{-i \int_0^\tau dt \frac{\dot{x}^2}{4}} \text{Pexp}\left\{ig \int_0^\tau dt (B_\mu^\Omega(x(t)) \dot{x}^\mu(t))\right\}
\]

\[
= \int \frac{d^4z}{4\pi^4} \delta(z_+) \frac{1}{(x-z)^2} \frac{\partial}{\partial z_-} \frac{1}{(z-y)^2} U^\Omega(z_\perp)
\]
Rotating back $B^\omega \rightarrow B$ we get

\[
(x\left| 1 \middle/ \mathcal{P}^2 \right| y) = \int \frac{d^4z}{4\pi^4} \delta(z_+) \frac{1}{(x - z)^2} \frac{\partial}{\partial z_-} \frac{1}{(z - y)^2} U(z_\perp; x, y),
\]

\[
U(z_\perp; x, y) = [x, z_x][z_x, z_y][z_y, y], \quad z_x = x_+e_- + z_-e_+ + z_\perp
\]
Propagators in the shock-wave background

Quark propagator

\[
(x | \frac{1}{\mathcal{P}} | y) = \int dz \delta(z_+) \frac{x - z'}{2\pi^2 (x - z)^4} \mathcal{P} + U(z_\perp; x, y) \frac{z' - y'}{2\pi^2 (z - y)^4}.
\]
Propagators in the shock-wave background

Quark-antiquark pair in a shock-wave background

\[
\text{Tr}\left\{ \gamma_{\mu} \left( x \ 1_{\mathcal{P}} \ y \right) \gamma_{\nu} \left( y \ 1_{\mathcal{P}} \ x \right) \right\} = - \int dz dz' \delta(z_+) \delta(z'_+) \\
\times \text{tr}\left\{ \gamma_{\mu} \frac{x - z}{2\pi^2 (x - z)^4} \phi + \frac{z - y}{2\pi^2 (z - y)^4} \gamma_{\nu} \frac{y - z'}{2\pi^2 (y - z')^4} \phi + \frac{z' - x}{2\pi^2 (z' - x)^4} \right\} U(z_\perp; z'_\perp)
\]

\[
U(z_\perp, z'_\perp) = \text{Tr}[z_x, z_y][z_y, z'_y][z'_y, z'_x][z'_x, z_x]
\]
The high-energy operator expansion is

\[(x - y)^4 T \{ \bar{\psi}(x) \gamma^\mu \psi(x) \bar{\psi}(y) \gamma^\nu \psi(y) \} = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu \nu}^{\text{LO}}(z_1, z_2) \text{tr}\{ \hat{U}_1^\eta \hat{U}_2^{\dagger \eta} \}\]

\[I_{\mu \nu}^{\text{LO}}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6 (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[ (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) - \frac{1}{2} \kappa^2 (\zeta_1 \cdot \zeta_2) \right].\]

\[\kappa \equiv \frac{1}{\sqrt{s} x^+} \left( \frac{p_1}{s} - x^2 p_2 + x_\perp \right) - \frac{1}{\sqrt{s} y^+} \left( \frac{p_1}{s} - y^2 p_2 + y_\perp \right)\]

\[\zeta_i \equiv \left( \frac{p_1}{s} + z_{i \perp} p_2 + z_{i \perp} \right), \quad \mathcal{R} \equiv \frac{\kappa^2 (\zeta_1 \cdot \zeta_2)}{2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)}\]
High-energy expansion in color dipoles

\[ Y > \eta \]
\[ Y < \eta \]

\( \eta \) - rapidity factorization scale

Step II - Evolution equation for color dipoles

\[
\frac{d}{d\eta} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2 (y - z)^2} \left[ \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \right] \\
- N_c \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \right) + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)
\]

(Linear part of \( K_{\text{NLO}} = K_{\text{NLO BFKL}} \))
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to $\eta_2$).

\[ \alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \otimes \]
Evolution equation in the leading order

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = K_{LO} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \ldots \Rightarrow \]

\[ \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle_{\text{shockwave}} = \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle_{\text{shockwave}} \]

\[ U_{z}^{ab} = \text{Tr}\{t^a U_z t^b U_\dagger_z\} \Rightarrow (U_x U_\dagger_y)^{\eta_1} \rightarrow (U_x U_\dagger_y)^{\eta_1} + \alpha_s (\eta_1 - \eta_2)(U_x U_\dagger_z U_z U_\dagger_y)^{\eta_2} \]

⇒ Evolution equation is non-linear
Derivation of the non-linear equation

The gluon propagator in a shock-wave external field in the $A_+ = 0$ gauge

$$
\langle \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \rangle
$$

$$
x_+ > 0 > y_+ \quad \Rightarrow \quad -\frac{i}{2} \int d^4z \, \delta(z) \, \frac{x_+ g_{\mu\xi} - e_{\mu}^+(x - z)_{\xi}^\perp}{\pi^2 [(x - z)^2 + i\epsilon]^2} \, U^{ab}_{z\perp} \frac{1}{\partial_+^{(z)}} \, \frac{y_+ \delta_{\nu}^{\perp\xi} - e_{\nu}^+(y - z)_{\xi}^\perp}{\pi^2 [(z - y)^2 + i\epsilon]^2}
$$

Diagram (a) = $g^2 \int_0^\infty dx_+ \int_{-\infty}^0 dy_+ \, \langle \hat{A}_\bullet^a, Y_1^1 (x_+, x_{\perp}) \hat{A}_\bullet^b, Y_1^1 (y_+, y_{\perp}) \rangle_{\text{Fig. (a)}}$

$$
= -4\alpha_s \int_0^{\epsilon^{\nu_1}} \frac{d\alpha}{\alpha} (x_{\perp}) \left| \frac{p_i}{p_{\perp}^2 - i\epsilon} U^{ab}_{\nu} \frac{p_i}{p_{\perp}^2 - i\epsilon} \right|_{y_{\perp}}
$$

$$
(x_{\perp} | F(p_{\perp}) | y_{\perp}) \equiv \int d^4p \, e^{i(p \cdot x - y)_{\perp}} F(p_{\perp}) \quad \text{- Schwinger’s notations}
$$
Derivation of the non-linear equation

Formally, the integral over $\alpha$ diverges at the lower limit, but since we integrate over the rapidities $Y > Y_2$ in the leading log approximation, we get ($\Delta Y \equiv Y_1 - Y_2$)

$$g^2 \int_0^\infty dx_+ \int_{-\infty}^0 dy_+ \langle \hat{A}_{s, Y_1}^a(x_+, x_\perp) \hat{A}_{s, Y_1}^b(y_+, y_\perp) \rangle \text{Fig.}(a) = -4\alpha_s \Delta Y (x_\perp \frac{P_i}{P_\perp} U^{ab} \frac{P_i}{P_\perp} |y_\perp)$$

$$\Rightarrow \langle \hat{U}_{z_1}^Y \otimes \hat{U}_{z_2}^{Y_1} \rangle \text{Fig.}(a) = -\frac{\alpha_s}{\pi^2} \Delta Y (t^a U_{z_1} \otimes t^b U_{z_2}^\dagger) \int d^2 z_3 \frac{(z_{13}, z_{23})}{z_{13}^2 z_{23}^2} U^{ab}_{z_3}$$

The contribution of the diagram in Fig. (b) is obtained by the replacement $t^a U_{z_1} \otimes t^b U_{z_2}^\dagger \rightarrow U_{z_1} t^b \otimes U_{z_2} t^a$, $z_2 \leftrightarrow z_1$. The two remaining diagrams(c) and (d) are obtained by $z_2 \rightarrow z_1$ for Fig.(c) and $z_1 \rightarrow z_2$ for Fig.(d).

Result:

$$\langle \text{Tr}\{ \hat{U}_{z_1}^Y \hat{U}_{z_2}^{Y_1} \} \rangle \text{Figs}(a)-(d) = \frac{\alpha_s \Delta Y}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \text{Tr}\{ t^a U_{z_1} U_{z_3}^\dagger t^a U_{z_2} U_{z_3}^\dagger \} - \frac{1}{N_c} \text{Tr}\{ U_{z_1} U_{z_2}^\dagger \} \right]$$
Derivation of the non-linear equation

Diagrams without the gluon-shockwave intersection:

These diagrams are proportional to the original dipole $\text{Tr}\{U_{z_1} U_{z_2}^\dagger\} \Rightarrow$ corresponding term can be derived from the contribution of Fig. (a)-(d) graphs using the requirement that the r.h.s. of the evolution equation should vanish in the absence of the shock wave ($U \to 1$).

$$\langle \text{Tr}\{\hat{U}_{z_1} Y_1 \hat{U}_{z_2}^\dagger Y_1\} \rangle = \frac{\alpha_s \Delta Y}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \text{Tr}\{t^a U_{z_1} U_{z_3}^\dagger t^a U_{z_3} U_{z_2}^\dagger\} - N_c \text{Tr}\{U_{z_1} U_{z_2}^\dagger\} \right]$$

$\Rightarrow$ non-linear equation for the evolution of the color dipole

$$\frac{d}{dY} \text{Tr}\{\hat{U}_{z_1} Y \hat{U}_{z_2}^\dagger Y\} = \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \text{Tr}\{\hat{U}_{z_1} Y \hat{U}_{z_3}^\dagger Y\} \text{Tr}\{\hat{U}_{z_1} Y \hat{U}_{z_2}^\dagger Y\} - N_c \text{Tr}\{\hat{U}_{z_1} Y \hat{U}_{z_2}^\dagger Y\} \right]$$
Non linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x)\hat{U}^\dagger(y)\} \]

BK equation

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z}{(x-z)^2(y-z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]

Non-linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x) \hat{U}^\dagger(y)\} \]

BK equation

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z}{(x-z)^2(y-z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]


LLA for DIS in pQCD \(\Rightarrow\) BFKL

(LLA: \(\alpha_s \ll 1, \alpha_s \eta \sim 1\))
Non-linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\} \]

BK equation

\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z}{(x-z)^2(y-z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]


LLA for DIS in pQCD ⇒ BFKL

(\text{LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1)

LLA for DIS in sQCD ⇒ BK eqn

(\text{LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1)

(s for semiclassical)
\[
\frac{d}{d\eta} \hat{U}(z_1, z_2) = \\
\frac{\alpha_s(\perp) N_c}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ \hat{U}(z_1, z_3) + \hat{U}(z_3, z_2) - \hat{U}(z_1, z_2) - \hat{U}(z_1, z_3)\hat{U}(z_3, z_2) \right\}
\]
Argument of coupling constant

\[ \frac{d}{d\eta} \hat{U}(z_1, z_2) = \]

\[ \frac{\alpha_s(\perp) N_c}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ \hat{U}(z_1, z_3) + \hat{U}(z_3, z_2) - \hat{U}(z_1, z_2) - \hat{U}(z_1, z_3) \hat{U}(z_3, z_2) \right\} \]

Renormalon-based approach: summation of quark bubbles

\[ -\frac{2}{3} n_f \rightarrow b = \frac{11}{3} N_c - \frac{2}{3} n_f \]
\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2} \int d^2 z \left[ \text{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\} \text{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c \text{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} \right] \]

\times \left[ \frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2} \left( \frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1 \right) + \frac{1}{z_{23}^2} \left( \frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1 \right) \right] + \ldots

\text{I.B.; Yu. Kovchegov and H. Weigert (2006)}

When the sizes of the dipoles are very different the kernel reduces to:

\[ \frac{\alpha_s(z_{12}^2)}{2\pi^2} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \quad |z_{12}| \ll |z_{13}|, |z_{23}| \]

\[ \frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \quad |z_{13}| \ll |z_{12}|, |z_{23}| \]

\[ \frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \quad |z_{23}| \ll |z_{12}|, |z_{13}| \]

⇒ the argument of the coupling constant is given by the size of the smallest dipole.
The nuclear modification factor is defined as:

$$R_{pPb}(p_T) = \frac{\frac{d^2N_{pPb}^{ch}}{d\eta dp_T}}{\langle T_{pPb} \rangle \frac{d^2\sigma_{pp}^{ch}}{d\eta dp_T}}$$

where $N_{pPb}^{ch}$ is the charged particle yield in p-Pb collisions.

ALICE arXiv:1210.4520

I. Balitsky (JLAB & ODU)

High-energy amplitudes and Wilson lines

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