Lifetime of hydrogen atom in classical electrodynamics

Suppose electron is rotating around nucleus in a circular orbit. For a circular orbit

$$\frac{mv^2}{R} = \frac{e^2}{4\pi\epsilon_0 R^2} \quad \Rightarrow \quad \frac{v^2}{R} = \frac{e^2}{4\pi\epsilon_0 m R^2}$$

The kinetic and potential energies are

$$E_{\rm kin} = \frac{mv^2}{2}, \qquad E_{\rm pot} = -\frac{e^2}{4\pi\epsilon_0 R}$$

so the total energy is

$$E = E_{\rm kin} + E_{\rm pot} = -\frac{e^2}{8\pi\epsilon_0 R}$$

The period of rotation is

$$T = \frac{2\pi R}{v} = \frac{4\pi^{3/2}}{|e|}\sqrt{\epsilon_0 m R}$$

The rotating electron radiates power

$$P = \mu_0 \frac{e^2 a^2}{6\pi c} \qquad - \quad \text{Larmor's formula}$$

Let us suppose that the loss of energy due to radiation over one period is much smaller that the total energy, then one can consider $R = R(t)$ slowly varying function of time.

After one revolution:

The radiated energy is

$$PT = \mu_0 \frac{e^2 a^2}{6\pi c} T = \mu_0 \frac{e^2}{6\pi c} \frac{v^2}{R^4} T$$

The change of $E = E_{\rm kin} + E_{\rm pot}$ is

$$E(R(t+T)) - E(R(t)) = -\frac{e^2}{8\pi\epsilon_0 R(t+T)} + \frac{e^2}{8\pi\epsilon_0 R(t)} \simeq \frac{e^2 T}{8\pi\epsilon_0 R^2(t)} \frac{dR(t)}{dt}$$

since $R(t+T) = R(t) + T\frac{dR(t)}{dt}$.

Thus, we get a differential equation

$$\frac{dR(t)}{dt} = -\frac{\mu_0 c e^4}{12\pi^2 m^2 R^2(t)} \quad \Leftrightarrow \quad R^2(t)\frac{dR(t)}{dt} = -\frac{\mu_0 c e^4}{12\pi^2 m^2} dt$$

Integrating this equation we get

$$\int R^2 dR = -\frac{\mu_0 c e^4}{12\pi^2 m^2}\int dt \quad \Rightarrow \quad \frac{R^3}{3} = -\frac{\mu_0 c e^4}{12\pi^2 m^2} t \ + \text{const}$$

Suppose at $t = 0$ $R = R_0$ (radius of the atom), then const $= \frac{R_0^2}{3}$ and we get

$$R_0^3 - R^3(t) = -\frac{\mu_0 c e^4}{4\pi^2 m^2} t$$

so the lifetime of the classical hydrogen atom is

$$\tau \;=\; \frac{4\pi^2 m^2}{\mu_0^2 c e^4} R_0^3$$

Taking $m = 9.1 \times 10^{-31}$kg, $e = 1.6 \times 10^{-19}$C, $\mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$ and estimating $R_0 \;\simeq$ Bohr radius $\simeq 5 \times 10^{-11}$m we get

$$\tau \;\simeq\; 1.3 \times 10^{-11}\mathrm{s}$$