I. TUNNELING THROUGH POTENTIAL BARRIER

Beam of free particles moving to the right is described by the wavefunction

$$\psi(x,t) \;=\; e^{ik(x-vt)} \;=\; e^{ikx-i\omega t} \;=\; e^{i\frac{p}{\hbar}x-i\frac{E}{\hbar}t}$$

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For the solution of stationary Schrödinger equation, e^{ikx} corresponds to right-moving particle whereas e^{-ikx} to the leftt-moving one.

Potential barrier: $V = V_0$ if $a > x > 0$ and $V = 0$ otherwise can be written as

$$V(x) \;=\; V_0\,\theta(a-x)\theta(x) \tag{1}$$

1. *Method of solution of stationary Schrödinger equation*

Stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x) \;=\; E\psi(x) \tag{2}$$

We will consider the case $V_0 > E$ (the case $E > V_0$ is similar).

$$\begin{aligned} -\tfrac{\hbar^2}{2m}\tfrac{d^2\psi(x)}{dx^2} &= E\psi(x) && x<0 \text{ and } x>a \\ -\tfrac{\hbar^2}{2m}\tfrac{d^2\psi(x)}{dx^2} &= -(V_0-E)\psi(x) && a\geq x\geq 0 \end{aligned} \tag{3}$$

Method of solution: solve in three separate regions and use matching condition: $\psi(x)$ and $\psi'(x)$ must be continuous at $x = 0$ and $x = a$

The equation (10) in the region $|x| < a$ can be rewritten as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} \;=\; E\psi(x) \quad\Leftrightarrow\quad \frac{d^2\psi(x)}{dx^2} \;=\; -\,\frac{2mE}{\hbar^2}\psi(x) \;=\; -\,k^2\psi(x) \tag{4}$$

(since $k = \frac{p}{\hbar}$). The general solution at $x < 0$ is

$$\psi_1(x) \;\stackrel{x\leq 0}{=}\; Ae^{ikx}+Be^{-ikx} \tag{5}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. This solution describes the sum of incident wave and reflected wave at $x < 0$.

The general solution in $a > x > 0$ region is

$$\psi_2(x) \stackrel{a>x>0}{=} Ce^{-\alpha x} + De^{\alpha x} \tag{6}$$

where $\alpha = \frac{\sqrt{2m(V_0-E)}}{\hbar}$.

The general solution in $x > a$ region can be written as

$$\psi_3(x) \stackrel{x\leq 0}{=} Fe^{ikx} + Ge^{-ikx} \tag{7}$$

However, if there is no left-moving beam incident on the barrier from the right we should set $G = 0$ (allowing $G \neq 0$ means that we consider an additional beam incident on our barrier from the right). Thus, in the region $x > a$ the solution has the form

$$\psi_3(x) \stackrel{x\leq 0}{=} Fe^{ikx} \tag{8}$$

The goal is to find the so-called transmission coefficient

$$T \equiv \frac{|F^2|}{|A|^2}$$

How to find T: use matching conditions.

2. Matching conditions

Matching conditions at $x = 0$:

$$\begin{aligned} \psi_1(0) = \psi_2(0) \quad &\Rightarrow \quad & A + B &= C + D \\ \psi_1'(0-) = \psi_2'(0+) \quad &\Rightarrow \quad & ik(A-B) &= -\alpha C + \alpha D \quad \Leftrightarrow \quad D - C = i\kappa(A-B) \end{aligned} \tag{9}$$

where $\kappa \equiv \frac{k}{\alpha}$.

Matching conditions at $x = a$:

$$\left.\begin{aligned} \psi_2(0) = \psi_3(0) \quad &\Rightarrow \quad & Ce^{-\alpha a} + De^{\alpha a} &= Fe^{ika} \\ \psi_2'(a-) = \psi_3'(a+) \quad &\Rightarrow \quad & -C\alpha e^{-\alpha a} + D\alpha e^{\alpha a} &= ikFe^{ika} \end{aligned}\right\} \Rightarrow \frac{De^{\alpha a} - Ce^{-\alpha a}}{De^{\alpha a} + Ce^{-\alpha a}} = i\kappa \Rightarrow C = \frac{1-i\kappa}{1+i\kappa}De^{2\alpha a} \tag{10}$$

Note that if you multiply all $A, B...F$ constants by the same number, nothing changes. To simplify calculations ,it is convenient to express A, B, C, D in terms of F rather than in terms of A. From Eq. (10) we get

$$\begin{aligned} C &= \frac{1-i\kappa}{2}e^{ika+\alpha a}F \\ D &= \frac{1+i\kappa}{2}e^{ika-\alpha a}F \end{aligned} \tag{11}$$

From Eq. (9) we get

$$
\begin{aligned}
A+B &= C+D = \frac{1}{2}[(1-i\kappa)e^{\alpha a}+(1+i\kappa)e^{-\alpha a}]e^{ika}F \\
A-B &= \frac{i}{\kappa}(C-D) = \frac{i}{2\kappa}[(1-i\kappa)e^{\alpha a}-(1+i\kappa)e^{-\alpha a}]e^{ika}F
\end{aligned}
\tag{12}
$$

and therefore

$$
A = \Big[\frac{1}{2}\Big(1+i\frac{1-\kappa^2}{2\kappa}\Big)e^{\alpha a}+\frac{1}{2}\Big(1-i\frac{1-\kappa^2}{2\kappa}\Big)e^{-\alpha a}\Big]e^{ika}F \tag{13}
$$

Inverting this expression we get

$$
F = \frac{1}{2}\Big[\Big(1+i\frac{1-\kappa^2}{2\kappa}\Big)e^{\alpha a}+\Big(1-i\frac{1-\kappa^2}{2\kappa}\Big)e^{-\alpha a}\Big]^{-1}e^{-ika}A \tag{14}
$$

and therefore the transmission coefficient is

$$
T \equiv \frac{|F|^2}{|A|^2} = \frac{4}{\Big[\frac{(1+\kappa^2)^2}{4\kappa^2}(e^{2\alpha a}+e^{-2\alpha a}-2)+4\Big]} = \Big[\frac{(1+\kappa^2)^2}{4\kappa^2}\sinh^2\alpha a+1\Big]^{-1} \tag{15}
$$

3. Result for transmisson coefficient

Since $\kappa^2=\frac{k^2}{\alpha^2}=\frac{E}{V_0-E}$ and $1+\kappa^2=\frac{V_0}{V_0-E}$ Eq. (15) can be rewritten as

$$
T = \Big[1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\alpha a\Big]^{-1} \tag{16}
$$

At large αa it can be approximated by

$$
T = 16\frac{E(V_0-E)}{V_0^2}e^{-2a\alpha} \tag{17}
$$

so the transmission coefficient is exponentially small.