Special relativity

Supplemental material

A. Galilean Transformations

Consider two inertial frames K, K', moving with a relative constant velocity $\vec{v}$. The coordinates in the two frames are related by

$$\begin{aligned} t' &= t \\ \vec{x}' &= \vec{x} - \vec{v}t \end{aligned} \tag{1}$$

Laws of mechanics are invariant under Galilean transformations. However, Maxwell's equations are *not* invariant. First people thought that Maxwell's equations are true only in the rest frame of "ether". Thus the natural question arose - *Is there a frame in which the "ether" is at rest"?*. Of course, we all know the answer (Michelson-Morley) that the velocity of light is the same in all frames, and the resolution of this paradox is the **Special Theory of Relativity**.

I. POSTULATES OF SPECIAL THEORY OF RELATIVITY

1. The same laws of nature hold in all systems moving uniformly with respect to one another.

2. The velocity of light has the same value in all systems moving uniformly with respect to each other, independent of velocity of observer relative to the source.

I. LORENTZ TRANSFORMATIONS

We will now derive the relationship between coordinates in two frames K, K' moving with constant velocity $\vec{v}$ relative to one another. For simplicity, we will let the origin of the coordinates coincide at $t = t' = 0$.

We suppose that a flashlight is rapidly switched on and off at the origin at $t = t' = 0$. Then, by postulate 2, observers in both K and K' see a spherical shell of radiation expanding with the velocity of light c. The wavefront satisfies

$$\text{In } K: \quad c^2t^2 - (x^2 + y^2 + z^2) = 0$$

$$\text{In } K': \quad c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0$$

Thus we see that, under such a transformation, the quantity $c^2t^2 - (x^2 + y^2 + z^2) = 0$ remains invariant. To consider the form of the transformations satisfying eqn. (3), we will specialize to the case where the axes in K, K' are parallel, and the frames are moving with a relative velocity $\vec{v} = v\vec{e}_z$. Because the transformations must reduce to Galilean transformations in the limit of small relative velocities, we need consider only the linear relations

$$t' = a_1 t + b_1 z$$

$$z' = a_2 t + b_2 z$$

$$x' = x$$

$$y' = y$$

Because the frames are moving with relative velocity v, we have that the event $x' = 0$ corresponds to $x = vt$, yielding

$$a_2 = -vb_2.$$

We now impose invariance of Δs^2:

$$c^2t^2 - (x^2 + y^2 + z^2) = c^2(a_1 t + b_1 z)^2 - (a_2 t + b_2 z)^2 - x^2 - y^2,$$

which we can expand as

$$c^2t^2[1 - a_1^2 + a_2^2/c^2] - z^2[1 + b - 1c^2 - b_2^2] + 2zt[a_2 b_2 - c^2 a_1 b_1] = 0.$$

This is true for all z, t, so equating the coefficients to zero yields

$$a_1^2 - a_2^2/c^2 = 1, \quad b_2^2 - c^2 b_1^2 = 1, \qquad a_2 b_2 = c^2 a_1 b_1.$$

Solving these simultaneous equations we find Lorentz transformations

$$\begin{aligned} ct' &= \gamma[ct - \tfrac{v}{c}z] \\ z' &= \gamma[z - \tfrac{v}{c}ct] \\ x' &= x \\ y' &= y \end{aligned} \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Now, using Lorentz transformations it is easy to see that the **interval** between two events

$$\Delta s^2 = c^2t^2 - (x^2 + y^2 + z^2)$$

is invariant under transformations between inertial frames.

Intervals:

$$\begin{aligned}
\Delta s^2 &= c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 > 0 \quad - \quad \text{timelike} \\
\Delta s^2 &= c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 < 0 \quad - \quad \text{spacelike} \\
\Delta s^2 &= c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0 \quad - \quad \text{light} - \text{like}
\end{aligned}$$

A. 4-vectors

It is convenient to define 4-vector of coordinate

$$x^{(0)} \equiv ct, \quad x^{(1)} \equiv x, \quad x^{(2)} \equiv y, \quad x^{(3)} \equiv z$$

(by convention, the indices of the 4-vector are upstairs). By definition, the square of the 4-vector is

$$x^2 \stackrel{\text{def}}{\equiv} (x^{(0)})^2 - (x^{(1)})^2 - (x^{(2)})^2 - (x^{(3)})^2 = c^2t^2 - \vec{x}^2$$

As we saw above, the square of the 4-vector of coordinate (= the interval between events $(0,\vec{0})$ and $(ct,\vec{x})$) is invariant under Lorentz transformations.

In general, a four-vector $(a^0,\vec{a})$ is defined as a combination of 4 numbers that are transformed as follows:

1. Under rotations: in a usual way

$$(a^{(0)})' = a^0, \quad \vec{a}' = \text{rotation of vector } \vec{a}$$

2. Under Lorentz boosts: like coordinate vector. For example, in the frame moving in x direction

$$\begin{aligned}(a^{(0)})' &= \gamma(a^{(0)} - \frac{v}{c}a^{(1)})\\ (a^{(1)})' &= \gamma(a^{(1)} - \frac{v}{c}a^{(0)})\\ (a^{(2)})' &= a^{(2)}, \qquad (a^{(3)})' = a^{(3)}\end{aligned}$$

The square of the four-vector is defined similarly to the interval = square of the four-vector of coordinate

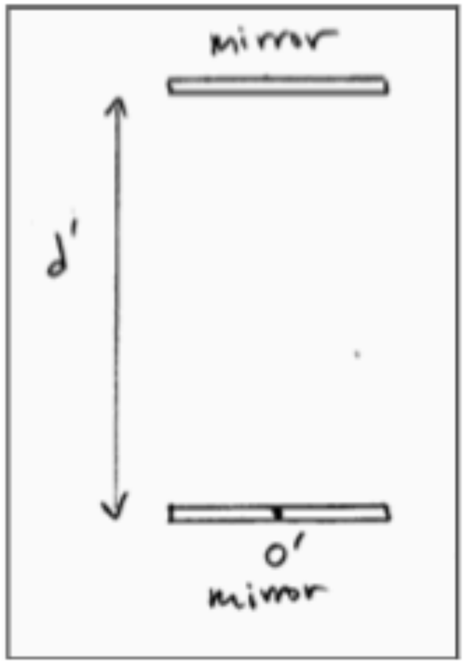
$$a^2 \stackrel{\text{def}}{\equiv} (a^{(0)})^2 - (a^{(1)})^2 - (a^{(2)})^2 - (a^{(3)})^2$$

It is easy to see that in both of these transformations (rotations and boosts)

$$a'^2 \equiv (a^{(0)'})^2 - (\vec{a}')^2 = (a^{(0)})^2 - \vec{a}^2 \equiv a^2$$

so the square of the four-vector is *invariant*. As we shall see later, the combination of energy and momentum forms a 4-vector of relativistic energy-momentum.

Consider a kind of clock:

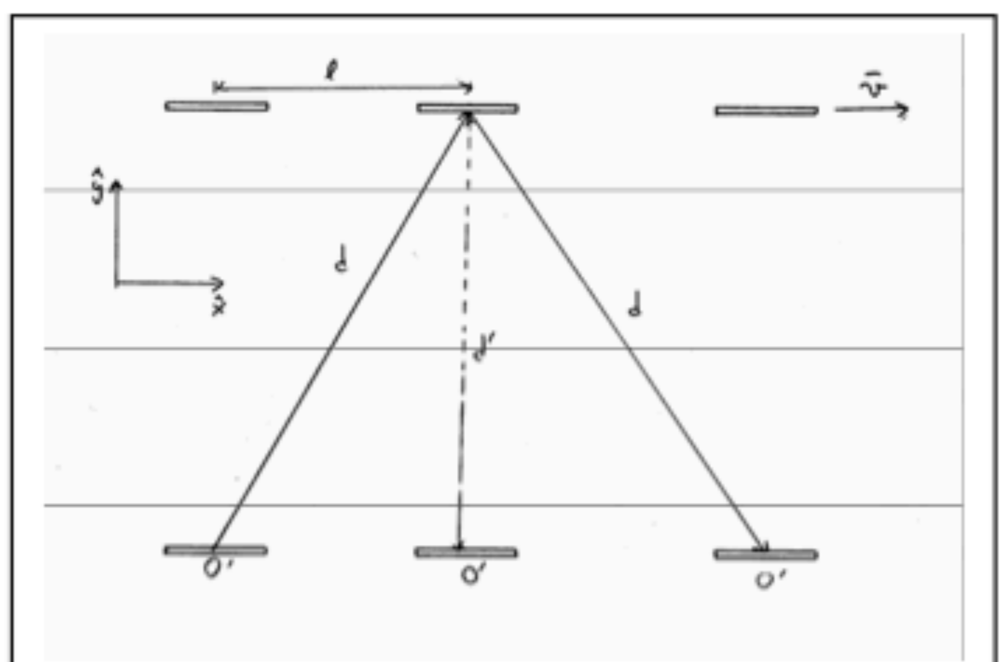


We observe two events: i) the emission of a flash at O' and ii) the reception of the flash at O'. In this case, $\Delta x' = \Delta y' = \Delta z' = 0$. The time interval between the two events is $\Delta t' = \frac{2d'}{c}$.

Now let's view the same two events from the point of view of another frame, S. As shown below, the S'-frame is moving to the right with speed v relative to the S-frame. In the S-frame, $\Delta x \neq 0$. The elapsed time is $\Delta t = \frac{2d}{c}$, where $d^2 = d'^2 + \ell^2$. Substitute for d, d', and ℓ in terms of Δt, $\Delta t'$, c, and v.

$$\frac{c^2 \Delta t^2}{4} = \frac{c^2 \Delta t'^2}{4} + \frac{v^2 \Delta t^2}{4}$$

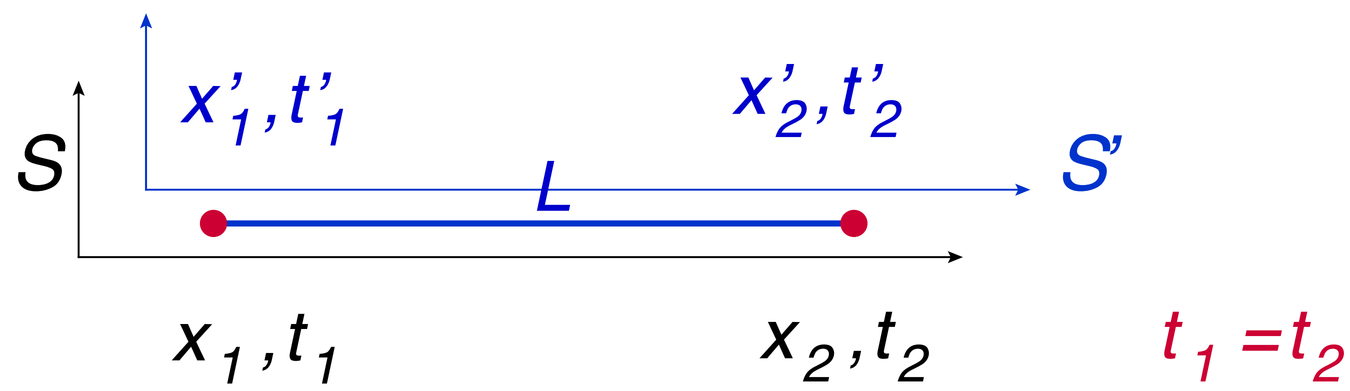
Solve for $\Delta t = \Delta t' \left(\frac{c^2}{c^2 - v^2} \right)^{1/2} = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}$.



Length contraction

Definition 1 (from the textbook):

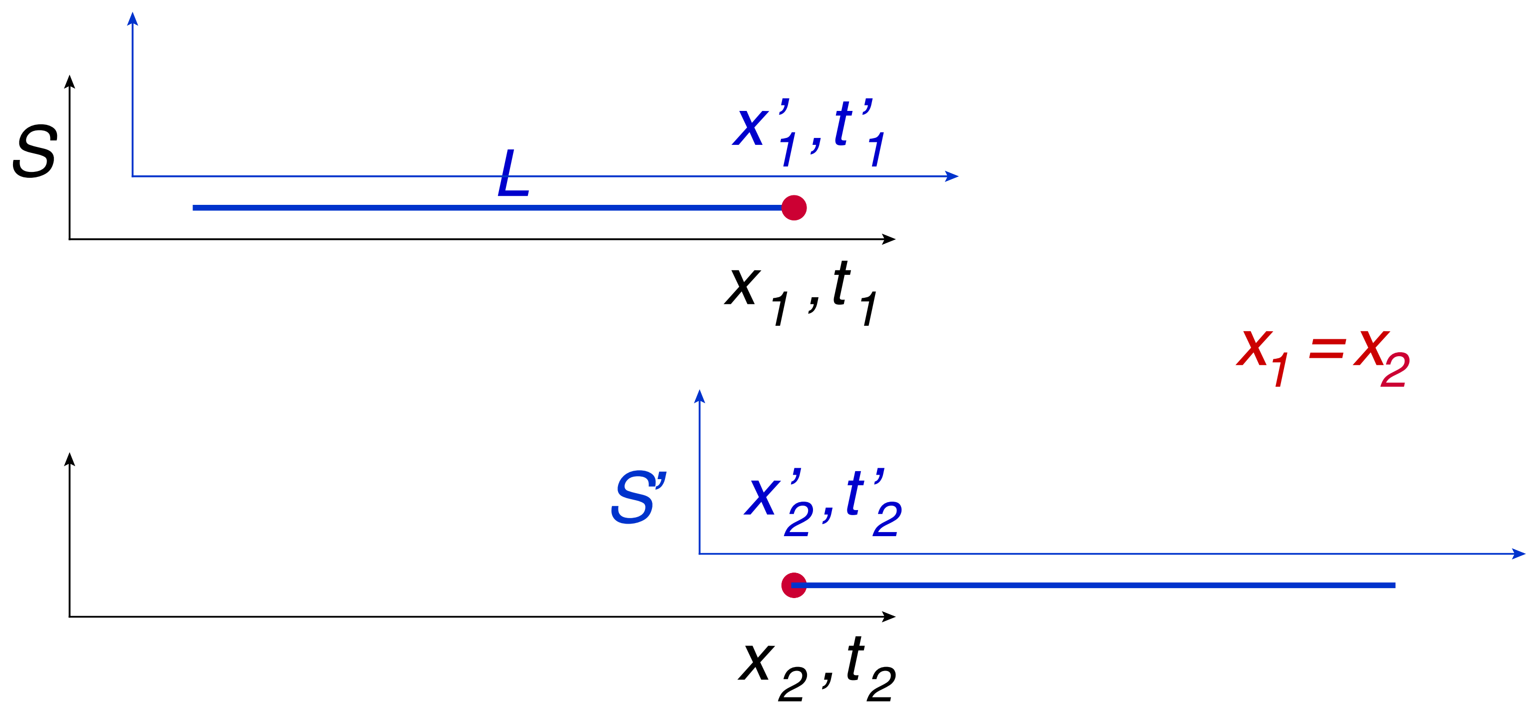
The length of the rod in frame S is defined as $L_S \equiv x_2 - x_1$, where x_2 is the position of one end at some time t_2 and x_1 is the position of the other end at the same time $t_1 = t_2$ as measured in frame S.



$$x_2' = \gamma(x_2 - vt_2), \quad x_1' = \gamma(x_1 - vt_1) \quad \Rightarrow \quad x_2 - x_1 = \frac{x_2' - x_1'}{\gamma} \quad \Rightarrow \quad L_S = \frac{L}{\gamma}$$

Definition 2:

At the same point $x_1 = x_2$ record $t_2 - t_1$ and multiply by v: $L_S \equiv v(t_2 - t_1)$



$$x_2' = \gamma(x_2 - vt_2), \quad x_1' = \gamma(x_1 - vt_1) \quad \Rightarrow \quad x_1' - x_2' \ = \ v\gamma(t_2 - t_1) \quad \Rightarrow \quad t_2 - t_1 \ = \ \frac{x_1' - x_2'}{v\gamma} \ = \ \frac{L}{v\gamma}$$

$$\Rightarrow \ L_S \ \equiv \ v(t_2 - t_1) \ = \ \frac{L}{\gamma}$$

Relativistic momentum

Suppose p = mv and check conservation of momentum in different frames

Suppose we on earth see 2 rocket ships passing one another.

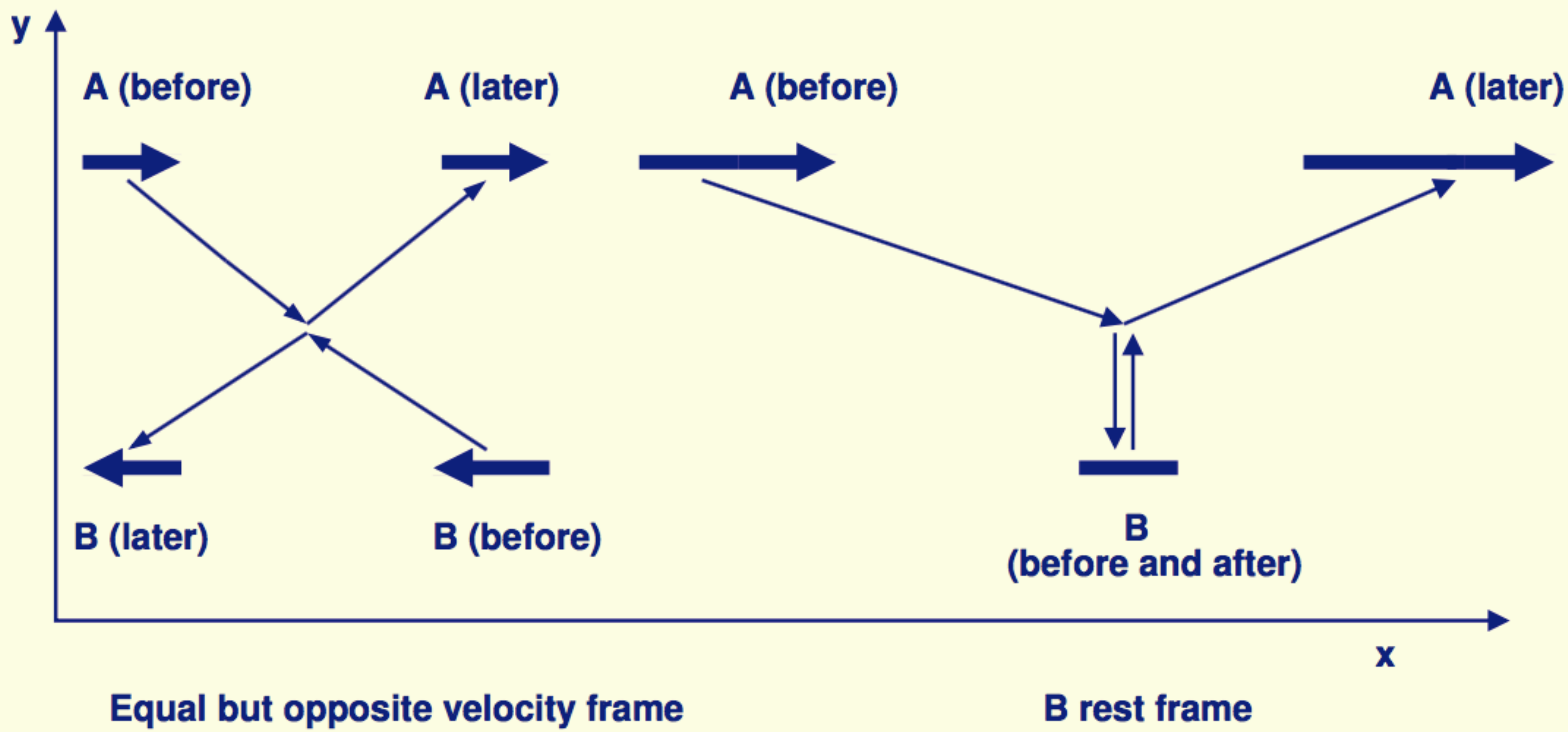


Figure 11: Depiction of two rockets passing one another: a) in equal but opposite velocity frame, each having velocity of magnitude v'; b) in B rest frame, where velocity of A appears to be v.

Someone on rocket A throws a ball down in y direction (in its frame) and someone on rocket B throws a ball up in y direction (in its frame).

By symmetry, balls collide elastically and go back to the respective starting points with exactly reversed velocities in y direction

$$\Delta u_y^A \ = \ -\Delta u_y^B \quad \Rightarrow \quad \Delta(m u_y^A) \ = \ -\Delta(m u_y^B)$$

- momentum conservation in Earth's frame.

In their respective frames each thinks he has thrown the ball with velocity of magnitude u, but from the viewpoint of the rocket B

$$|u_A^y| \ = \ \left|\frac{\Delta y_A}{\Delta t}\right| \ = \ \left|\frac{\Delta y'_A}{\gamma \Delta t'}\right| \ = \ \left|\frac{1}{\gamma} {u'}_y^A\right| \ \neq \ |u_B^y| \quad \Rightarrow \quad \Delta(m u_y^A) \ \neq \ -\Delta(m u_y^B)$$

- no momentum conservation in the frame of rocket B.

$$\text{Way out : define momentum as} \quad \vec{p} \ \stackrel{\text{def}}{\equiv} \ \gamma m \vec{u} \ = \ \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

The momentum thus defined is conserved in all inertial frames

Problem about CEBAF upgrade

In CEBAF accelerator at JLab, the electrons with energy E = 6GeV scatter from the proton target which is at rest.

1. Is it possible to produce charmed particles in CEBAF accelerator?
2. What about the upgraded CEBAF with the electron energy 12 GeV?

The lightest charmed particle J/ψ has a rest mass 3.1 GeV.

Hint: When we have barely enough energy, the electron, the proton and the produced J/ψ particle have negligible relative velocities in the final state.

Necessary information

Lorentz transformations for energy and momentum (frame K' moves in positive x direction with velocity v)

$$E' = \gamma(E - vp_x)$$

$$p'_x = \gamma\left(p_x - \frac{v}{c^2}E\right)$$

$$p'_y = p_y, \quad p'_z = p_z$$

These laws of transformation are the same as for the 4-vector of coordinate $\Rightarrow$ the combination $\mathcal{P} = (E, \vec{p}c)$ forms a 4-vector of relativistic momentum. It is easy to check that $\mathcal{P}^2 = E^2 - p^2c^2$ does not change under the above Lorentz transformations. Also, because both E and $\vec{p}$ are conserved in the collision, the four-vector is conserved: $(\mathcal{P})_{\text{before}} = (\mathcal{P})_{\text{after}}$.

Solution

I will use units where $c = 1$, then mass of the proton is 940 MeV, mass of the electron is ~ 0.5 MeV, and mass of the J/ψ particle is 3.1 GeV. Denote the 4-momenta of electron and proton before the collision by $\mathcal{P}_e$ and $\mathcal{P}_p$, and 4-momenta of electron, proton, and J/ψ particle after the collision by $\mathcal{K}_e$, $\mathcal{K}_p$, and $\mathcal{K}_J$, respectively.

Due to momentum conservation

$$\begin{aligned} \mathcal{P}_e + \mathcal{P}_p &= \mathcal{K}_e + \mathcal{K}_p + \mathcal{K}_J \\ \Rightarrow \ (\mathcal{P}_e + \mathcal{P}_p)^2 &= (\mathcal{K}_e + \mathcal{K}_p + \mathcal{K}_J)^2 \qquad (*) \end{aligned}$$

Since $(\mathcal{P}_e + \mathcal{P}_p)^2 = (\mathcal{K}_e + \mathcal{K}_p + \mathcal{K}_J)^2$ is Lorentz invariant we can calculate it in any frame.
The l.h.s in the lab frame is

$$\begin{aligned} (\mathcal{P}_e + \mathcal{P}_p)^2 &= m_e^2 + m_p^2 + 2p_e^0 p_p^0 - 2\vec{p}_e \cdot \vec{p}_p \\ &= m_e^2 + m_p^2 + 2p_e^0 p_p^0 = m_e^2 + m_p^2 + 2E_e m_p \end{aligned}$$

where $E_e = 6$ GeV currently and $E_e \stackrel{\text{will be}}{=} 12$ GeV for the upgraded CEBAF.

As to the r.h.s. of Eq. $(*)$,
it is convenient to consider it in the c.m. frame

$$(\mathcal{K}_e + \mathcal{K}_p + \mathcal{K}_J)^2 \;=\; (E_k + E_p + E_J)^2 - (\vec{k}_e + \vec{k}_p + \vec{k}_J) \;\geq\; (E_e + E_p + E_J)^2$$
$$=\; (\sqrt{k_e^2 + m_e^2} + \sqrt{k_p^2 + m_p^2} + \sqrt{k_J^2 + m_J^2} \;\geq\; (m_e + m_p + m_J)^2$$

Thus, we get

$$m_e^2 + m_p^2 + 2E_e m_p \;\geq\; (m_e + m_p + m_J)^2 \;\simeq\; 16\mathrm{GeV}^2$$

At present beam energy $m_e^2 + m_p^2 + 2E_e m_p \sim 13\mathrm{GeV}^2$ so J/ψ particle cannot be produced while for the upgraded CEBAF $m_e^2 + m_p^2 + 2E_e m_p \sim 25\mathrm{GeV}^2$ which permits creation of the J/ψ particle.