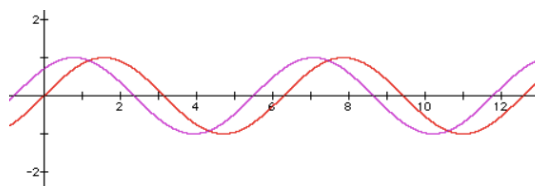
I. WAVES

A. Types of waves

Mathematically, the most basic wave is the (spatially) one-dimensional sine wave (also called harmonic wave or sinusoid) with an amplitude described by the equation:

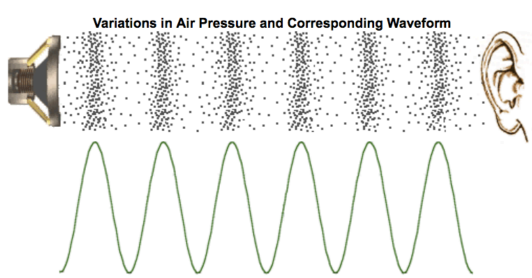
$$u(x,t) \;=\; A\sin[kx-\omega t+\delta] \;=\; A\cos[kx-\omega t+\delta-\frac{\pi}{2}] \tag{1}$$

A - "amplitude", $kx-\omega t+\delta =$ "phase", $\delta =$ "phase constant" or "phase shift"



If $u(x,t)=y(x,t)$ this equation describes a "transverse" wave moving to the right with velocity v.

Mechanical transverse waves correspond to situation when displacement y is orthogonal to the direction of motion of the wave, like in a string. Longitudinal waves cause the medium to vibrate parallel to the direction of the wave. It consists of multiple compressions and rarefactions. Example- sound waves.



Electromagnetic wave

$$\vec{E}(\vec{r},t) \;=\; A\vec{e}(\vec{k})\sin(\vec{k}\cdot\vec{r}-\omega t+\delta), \qquad \vec{B}=\frac{\hat{k}}{c}\times\vec{E}$$

Here $\omega = ck$ and $\vec{e}(\vec{k})$ - "polarization vector", $\vec{k}\cdot\vec{e}(\vec{k})=0$.

B. Wave equation

Eq. (1) describes sine wave moving to the right with velocity $v = \frac{\omega}{k}$.

$$u(x,t) \;=\; A\sin[kx - \omega t + \delta] \;=\; A\sin[k(x - vt) + \delta] \tag{2}$$

Left-moving wave has the form

$$u(x,t) \;=\; A\sin[k(x + vt) + \delta] \tag{3}$$

Superposition of right- and left-moving waves with same amplitudes is a standing wave

$$A\sin[k(x - vt)] + A\sin[k(x + vt)] \;=\; 2A\sin kx \cdot \cos kvt \;=\; 2A\sin kx \cdot \cos\omega t$$

In general, a wave is a solution of a wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} \;=\; \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{4}$$

It is easy to check that

$$F(x - vt) + G(x + vt)$$

with arbitrary F and G is a solution of this equation. Such solution is a superposition of many (or infinitely many) sine waves with different k's.

C. Description of waves with complex exponentials

Sometimes is very convenient to describe a (right-moving) sine wave

$$a\cos[kx - \omega t + \delta] \tag{5}$$

as a real part of the expression

$$u(x,t) \;=\; Ae^{i(kx - \omega t + \delta)} \tag{6}$$

with complex amplitude $A = ae^{i\delta}$. (Recall that $e^{i\phi} = \cos\phi + i\sin\phi$.) The "physical wave" is then

$$\Re u(x,t) \;=\; \Re Ae^{i(kx-\omega t)} \;=\; \Re ae^{i(kx-\omega t+\delta)} \;=\; a\cos[kx - \omega t + \delta] \tag{7}$$

The left-moving wave with $\omega = vk$ is described similarly by

$$\Re u(x,t) \;=\; \Re Ae^{i(-kx-\omega t)} \;=\; \Re ae^{i(-kx-\omega t+\delta)} \;=\; a\cos[kx + \omega t - \delta] \tag{8}$$

D. Wave packets and group velocity

Let us consider the simple case of two monochromatic waves, of the same amplitude and of neighbouring frequencies (k_1, ω_1) and (k_2, ω_2), where $k_1, k_2 \sim k_0$. Then the resulting "wave packet" propagates as

$$
\begin{aligned}
U(x,t) &= A\left[e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}\right] \\
&= A e^{i[(k_1+k_2)x/2 - (\omega_1+\omega_2)t/2]} \left\{e^{i[(k_1-k_2)x/2-(\omega_1-\omega_2)t/2]} + e^{i[(k_2-k_1)x/2+(\omega_2-\omega_1)t/2]}\right\} \\
&= 2A\cos\left[\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right] e^{i[(k_1+k_2)x/2-(\omega_1+\omega_2)t/2]}
\end{aligned}
$$

We have written the wave as a *slowly moving amplitude factor* with velocity

$$v_g = \frac{\omega_1 - \omega_2}{k_1 + k_2} \longrightarrow \left.\frac{d\omega}{dk}\right|_{k_0} \quad \text{as } k_2 \to k_1, \tag{9}$$

known as the $\boxed{\textbf{group velocity}}$, and a rapidly moving "phase" with velocity

$$v_p \longrightarrow \frac{\omega_1+\omega_2}{k_1+k_2} = \frac{\omega}{k} \quad \text{as } k_2 \to k_1. \tag{10}$$

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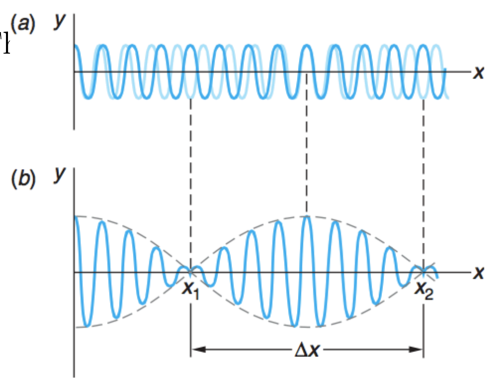

Figure 5-15 Two waves of slightly different wavelength and frequency produce beats. (*a*) Shows $y(x)$ at a given instant for each of the two waves. The waves are in phase at the origin, but because of the difference in wavelength, they become out of phase and then in phase again. (*b*) The sum of these waves. The spatial extent of the group Δx is inversely proportional to the difference in wave numbers Δk, where k is related to the wavelength by $k = 2\pi/\lambda$. Identical figures are obtained if y is plotted versus time t at a fixed point x. In that case the extent in time Δt is inversely proportional to the frequency difference $\Delta\omega$.

E. "Gaussian hat"

Definition:

$$g(x) \;=\; \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-\frac{x^2}{4\Delta x^2}}$$

Normalization: since

$$\int_{-\infty}^{\infty} dx\; e^{-ax^2} \;=\; \sqrt{\frac{\pi}{a}} \qquad - \quad \text{“Gauss' integral”}$$

we have

$$\int_{-\infty}^{\infty} dx\; |g(x)|^2 \;=\; 1$$

Why the notation is Δx^2:

$$\langle x^2\rangle \equiv \int_{-\infty}^{\infty} dx\ x^2 g^2(x) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4}\int_{-\infty}^{\infty} dx\ x^2 e^{-\frac{x^2}{2\Delta x^2}} = (\Delta x)^2 \equiv \Delta x^2$$

where we used $\int_{-\infty}^{\infty} dx\ e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$.

Right-moving Gaussian wave packet:

$$g(x-vt) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-\frac{(x-vt)^2}{4\Delta x^2}}$$

It is a superposition of waves with infinitely many k's.

II. FOURIER TRANSFORMATION

Fourier transformation:

$$\bar{f}(k) \equiv \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\ e^{-ikx} f(x)$$

Inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\ e^{ikx}\bar{f}(k)$$

Example: Fourier transformation of Gaussian hat

$$\bar{g}(k) \equiv \int\frac{dx}{\sqrt{2\pi}}\ e^{-ikx}g(x) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4}\int\frac{dx}{\sqrt{2\pi}}\ e^{-ikx}e^{-\frac{x^2}{4\Delta x^2}} = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4}\int\frac{dx}{\sqrt{2\pi}}\ e^{-\frac{x^2+4ikx\Delta x^2}{4\Delta x^2}}$$
$$= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-k^2\Delta x^2}\int\frac{dx}{\sqrt{2\pi}}\ e^{-\frac{(x+2ik\Delta x^2)^2}{4\Delta x^2}} = \left(\frac{1}{2\pi\Delta k^2}\right)^{1/4} e^{-\frac{x^2}{4\Delta k^2}} \tag{11}$$

where $\Delta k = \frac{1}{2\Delta x}$. Thus, Fourier transform of a Gaussian hat is a Gaussian hat again with

$$\Delta k\Delta x = \frac{1}{2} \qquad (*)$$

Similarly, if one considers a time-dependent Gaussian hat

$$h(t) = \left(\frac{1}{2\pi\Delta t^2}\right)^{1/4} e^{-\frac{t^2}{4\Delta t^2}}$$

its Fourier transform is a Gaussian hat in frequency

$$\tilde{h}(\omega) \equiv \int\frac{dt}{\sqrt{2\pi}}\ e^{i\omega t}h(t) = \left(\frac{1}{2\pi\Delta\omega^2}\right)^{1/4} e^{-\frac{t^2}{4\Delta\omega^2}}, \qquad \Delta\omega \equiv \frac{1}{2\Delta t}$$

so

$$\Delta\omega\Delta t = \frac{1}{2} \qquad (**)$$

These properties ($(*)$ and $(**)$) are called *classical uncertainty relations.*

III. PROPAGATION OF A GAUSSIAN WAVE PACKET IN THE DISPERSIVE MEDIUM

Waves propagating in a dispersive medium have non-linear dependence $\omega = \omega(k)$ (and linear dependence $\omega = vk$) corresponds to a non-dispersive medium). Let us consider propagation of a Gaussian wave packet in a dispersive medium.

First, let us recall the propagation of a Gaussian pulse in a linear medium without dispersion

$$u_0(x,t) \;=\; \Big(\frac{1}{\pi L^2}\Big)^{1/4} \;\exp\left\{-\frac{(x-vt)^2}{2L^2} + ik_0(x-vt)\right\} \tag{12}$$

where $L = \Delta x\sqrt{2}$ is the width of the Gaussian wave packet.

Suppose at $t = 0$ we switch on the dispersion so that $\omega = \omega(k)$ (some non-linear function). What will happen with the pulse? For simplicity, let us consider an approximate model of the behavior of frequency in the vicinity of ω_0 in the form

$$\omega(k) \;=\; \omega_0\left(1 + \frac{a^2k^2}{2}\right) \tag{13}$$

where $\omega_0 = vk_0$ is the center of our Gaussian wave packet.

We obtain after some math

$$u(x,t) = \Re\frac{2(4\pi L^2)^{1/4}}{\sqrt{L^2 + i\omega_0 a^2 t}} e^{-i\omega_0 t\left(1+\frac{a^2k_0^2}{2}\right)+ik_0x} \;\exp\left\{-\frac{(x-\omega_0 a^2 k_0 t)^2}{2L^2(1+i\omega_0\frac{a^2t}{L^2})}\right\} \tag{14}$$

The peak of the pulse (14) is located at $x = \omega_0 a^2 k_0 t \Rightarrow$ it moves with the *group velocity* $\frac{\partial\omega_k}{\partial k}\Big|_{k=k_0} = \omega_0 a^2 k_0$.

The wave packet spreads as it moves:

$$\sqrt{2}\Delta x(t) \equiv L(t) = \sqrt{L^2 + \frac{a^4\omega_0^2 t^2}{L^2}}$$

This is a general feature of non-linear Gaussian wave packets: for the same reason ($\omega_k = \sqrt{(m^2c^4/\hbar^2) + k^2}$) wave packets corresponding to relativistic particles broaden with time.

IV. INTERFERENCE PATTERN

A. Intensity of a superposition of waves

Intensity of any wave $\Phi(x,t)$ is defined as time average of energy

$$I(x) \overset{\text{def}}{\equiv} C \times \lim_{T\to\infty}\int_0^T dt\ \Phi^2(t)$$

The constant C depends on physics, e.g. for plane electromagnetic wave

$$\vec{E}(x,t) \ = \ \hat{e}_y E_0 \cos(\omega t - kx)],\quad \vec{B} = \hat{e}_z \frac{E_0}{c}\cos(\omega t - kx)]$$

the energy is $\frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \epsilon_0 E^2$ so the constant $C = \epsilon_0$. For simplicity, we will take $C = 2$ in what follows.

Consider sum of plane waves with different phase shifts

$$\Phi(t,x) \ = \ a_1\cos(\omega t - kx + \delta_1) + a_2\cos(\omega t - kx + \delta_2)$$

For simlicity, take $x = 0$ (for $x \neq 0$ you can always absorb kx into phase shift)

1. *Intensity for superposition of waves: in terms of real numbers*

$$I \ = \ 2\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \{a_1^2\cos^2(\omega t + \delta_1) + a_2^2\cos^2(\omega t + \delta_2) + 2a_1a_2\cos(\omega t + \delta_1)\cos(\omega t + \delta_2)\}$$
$$= \ 2\lim_{T\to\infty}\int_0^T \frac{dt}{T}\ \{a_1^2\cos^2(\omega t + \delta_1) + a_2^2\cos^2(\omega t + \delta_2) + a_1a_2[\cos(\delta_{12}) + \cos(2\omega t + \delta_1 + \delta_2\}$$

Property

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \ \cos(\omega t + \phi) \ = \ \lim_{T\to\infty}\frac{\sin(\omega T + \phi) - \sin\phi}{\omega T} \ = \ 0$$

Corollary

$$\lim_{T\to\infty}\int_0^T dt\ \ \cos^2(\omega t + \phi) \ = \ \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \frac{1 + \cos 2(\omega t + \phi)}{2} \ = \ \frac{1}{2}$$

We get

$$I \ = \ 2\lim_{T\to\infty}\int_0^T \frac{dt}{T}\ \{a_1^2\cos^2(\omega t + \delta_1) + a_2^2\cos^2(\omega t + \delta_2) + a_1a_2[\cos\delta_{12} + \cos(2\omega t + \delta_1 + \delta_2)\}$$
$$= \ 2[\frac{a_1^2}{2} + \frac{a_2^2}{2} + a_1a_2\cos(kr_{12} + \delta_{12}) + 0 \ = \ a_1^2 + a_2^2 + 2a_1a_2\cos(kr_{12} + \delta_{12})$$

2. Intensity for superposition of waves: in terms of complex numbers

Description of waves in terms of complex numbers

$$\Psi(y,t) \;=\; A_1 e^{i\omega t} + A_2 e^{i\omega t},$$

where $A_1 = a_1 e^{i\delta_1}$, $A_2 = a_2 e^{i\delta_2}$

Relation between complex Ψ and real Φ

$$\Phi(y,t) \;=\; \Re\Psi(y,t)$$

Formula for intensity in terms of complex wave

$$I \;=\; \frac{C}{2}\lim_{T\to\infty}\int_0^T dt\;|\Psi^2(t)|$$

C=2 inour example so

$$I \;=\; \lim_{T\to\infty}\int_0^T dt\;|\Psi^2(t)|$$

Proof:

$$\begin{aligned}
\lim_{T\to\infty}\int_0^T dt\;|\Psi^2(t)| \;&=\; \lim_{T\to\infty}\int_0^T dt\;[A_1^* e^{-i\omega t} + A_2^* e^{-i\omega t}][A_1 e^{i\omega t} + A_2 e^{i\omega t}] \\
=\; \lim_{T\to\infty}\int_0^T dt\;(|A_1|^2 + |A_2|^2 + A_1^* A_2 + A_1 A_2^*) \;&=\; (|A_1|^2 + |A_2|^2 + A_1^* A_2 + A_1 A_2^*) \\
=\; (a_1^2 + a_2^2 + a_1 a_2 e^{i\delta_{12}} + a_1 a_2 e^{-i\delta_{12}}) \;&=\; a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta_{12}
\end{aligned}$$

B. Double-slit interference pattern

Spherical wave

$$\Psi(t,r) \;=\; A e^{i(kr-\omega t)} \qquad \Leftrightarrow \qquad \Phi(r,t) \;=\; a\cos(kr - \omega t + \delta)$$

Due to Huygens' principle, each of the slits can be considered as a source of spherical waves. Since slits are symmetric (w.r.t. light source) $\delta_1 = \delta_2$ so we can take $\delta_1 = \delta_2 = 0$ (in complex description, $A_1 = A_2 = A$ and A is real).

Description of superposition of waves in terms of real numbers

$$\Phi(y,t) \;=\; \Phi_1(y,t) + \Phi_2(y,t) \;=\; a\cos(\omega t - kr_1) + a\cos(\omega t - kr_2)$$

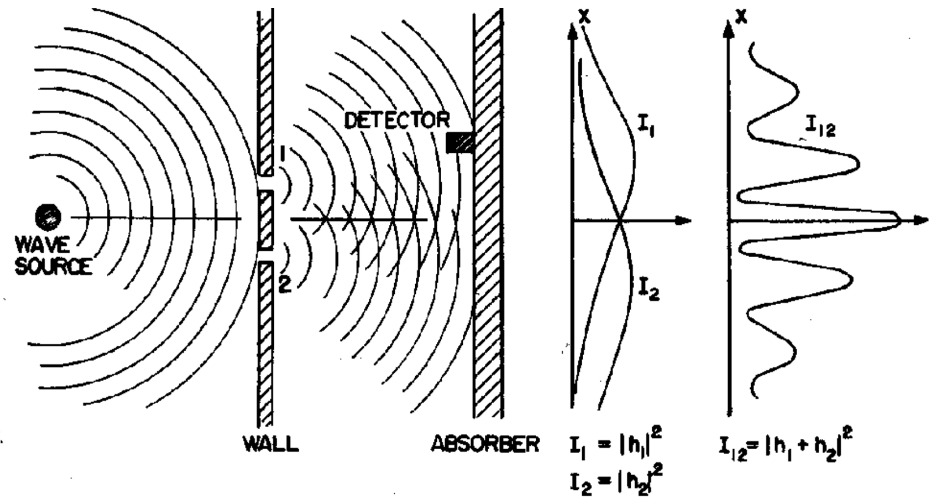

FIG. 1. Fig.1-2 from *Feynman Lectures on Physics*, v.3

For our wave ($r_{12} \equiv r_1 - r_2$)

$$
\begin{aligned}
I &= 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \; \{a^2 \cos^2(\omega t - kr_1) + a^2 \cos^2(\omega t - kr_2) + 2a^2 \cos(\omega t - kr_1)\cos(\omega t - kr_2)\} \\
&= 2 \lim_{T\to\infty} \int_0^T \frac{dt}{T} \; \{a^2 \cos^2(\omega t - kr_1) + a^2 \cos^2(\omega t - kr_2) + a^2[\cos kr_{12} + \cos[2\omega t - k(r_1 + r_2)]\} \\
&= a_1^2 + a_2^2 + 2a_1 a_2 \cos kr_{12}
\end{aligned}
$$

Superposition in terms of complex waves

$$\Psi(y,t) \;=\; \Psi_1(y,t) + \Psi_2(y,t) \;=\; ae^{i(kr_1-\omega t)} + ae^{i(kr_2-\omega t)} \;=\; ae^{-i\omega t}(e^{ikr_1} + e^{ikr_2})$$

We get

$$I \;=\; \lim_{T\to\infty} \int_0^T \frac{dt}{T} \; |\Psi^2(y,t)| \;=\; a^2|e^{ikr_1} + e^{ikr_2}|^2 \;\equiv\; |\Psi(y)|^2 \;=\; |\Psi_1 + \Psi_2|^2$$

so intensity of the superposition of the two waves is

$$I \;=\; |\Psi_1|^2 + |\Psi_2|^2 + (\Psi_1^*\Psi_2 + \Psi_1\Psi_2^*) \;=\; I_1 + I_2 + \text{interference term}$$