Problem 3.1 *The Bethe–Weizsäcker formula of Eq. (3.5) provide an excellent representation of the mass systematics of nuclei. Sho explicitly that, for fixed A, M(A, Z) has a minimum value. Is this evidence for the "valley of stability" observed in Fig. 2.3? What the stablest nucleus with A = 16? What about A = 208? (You ca differentiate Eq. (3.5), or simply plot M as a function of Z.)*

From Eq. (3.5) of the text, the mass of the nucleus as a function o its charge (Z) and the nucleon number (A) is given by

$$M(A,Z) = (A-Z)m_n + Zm_p - \frac{a_1}{c^2}A + \frac{a_2}{c^2}A^{\frac{2}{3}} + \frac{a_3}{c^2}\frac{Z^2}{A^{\frac{1}{3}}} + \frac{a_4}{c^2}\frac{(A-2Z)^2}{A} \pm a_5 A^{-\frac{3}{4}}, \tag{3.1}$$

where the values of the positive coefficients $a_1, \ldots, a_5$ are given in Eq. (3.4) of the text. For a fixed value of A, the nuclear mass becomes a function of its charge alone. The value of the nuclear charge for which the mass becomes a minimum can be easily determined from

$$\left.\frac{\partial M(A,Z)}{\partial Z}\right|_{\text{fixed } A} = -m_n + m_p + \frac{2a_3 Z}{c^2 A^{\frac{1}{3}}} - \frac{4a_4(A-2Z)}{c^2 A} = 0$$

$$\text{or} \quad \frac{2Z}{c^2 A}\left(a_3 A^{\frac{2}{3}} + 4a_4\right) = \frac{1}{c^2}\left(4a_4 + (m_n - m_p)c^2\right) \tag{3.2}$$

$$\text{or} \quad Z_{\text{min}}(A) = \frac{A}{2} \times \frac{4a_4 + (m_n - m_p)c^2}{4a_4 + a_3 A^{\frac{2}{3}}}.$$

To determine whether this stationary point is a minimum or a maximum, we note that

$$\left.\frac{\partial^2 M(A,Z)}{\partial^2 Z}\right|_{\text{fixed } A} = \frac{2}{c^2 A}\left(4a_4 + a_3 A^{\frac{2}{3}}\right) > 0. \tag{3.3}$$

We therefore conclude that the extremum in Eq. (3.2) is a minimum.

The solution for the minimum in Eq. (3.2) represents a valley of minima, similar to what is observed in Fig. 2.3 of the text. Noting from Eqs. (2.2) and (3.4) of the text that

$$a_3 = 0.72\,\text{MeV}, \quad a_4 = 23.3\,\text{MeV}, \quad (m_n - m_p)c^2 \approx 1.29\,\text{MeV}, \tag{3.4}$$

we conclude that for low values of A, the minimum in the mass occurs for

$$Z_{\text{min}} \approx \frac{A}{2}, \tag{3.5}$$

which represents the stability line. However, as A increases, the denominator becomes larger than the numerator and we have

$$Z_{\text{min}} < \frac{A}{2} \quad \text{or} \quad N > Z_{\text{min}}, \tag{3.6}$$

where we have used the fact that $A = Z + N$. This deviation from stability starts when

$$a_3 A^{\frac{2}{3}} > (m_n - m_p)c^2$$

$$\text{or} \quad A > \left(\frac{(m_n - m_p)c^2}{a_3}\right)^{\frac{3}{2}} \approx \left(\frac{1.29\,\text{MeV}}{0.72\,\text{MeV}}\right)^{\frac{3}{2}} \approx 2.2, \tag{3.7}$$

which is somewhat low compared to the experimental observations sketched in Fig. 2.3 of the text.

From the formula in (3.2) we can calculate

$$Z_{\text{min}}(A=16) = \frac{A}{2} \times \frac{4a_4 + (m_n - m_p)c^2}{4a_4 + 0.72A^{\frac{2}{3}}}$$
$$= \frac{16}{2} \times \frac{4 \times 23.3\,\text{MeV} + 1.29\,\text{MeV}}{4 \times 23.3\,\text{MeV} + 0.72\,\text{MeV} \times (16)^{\frac{2}{3}}}$$
$$\approx 8 \times 0.95 = 7.6, \tag{3.8}$$
$$Z_{\text{min}}(A=208) = \frac{208}{2} \times \frac{4 \times 23.3\,\text{MeV} + 1.29\,\text{MeV}}{4 \times 23.3\,\text{MeV} + 0.72\,\text{MeV} \times (208)^{\frac{2}{3}}}$$
$$\approx 104 \times 0.79 \approx 82.16.$$

We note that for $A = 16$ and $A = 208$, we expect $^{16}O^{8}$ and $^{208}Pb^{82}$, respectively, to represent the most stable nuclei, and so this model leads to reasonably accurate predictions.

This result can also be obtained graphically as follows. First, we note that, for a fixed value of A, we can write the mass formula in Eq. (3.1) as

$$M(A,Z)c^2 = \frac{4a_4 + a_3 A^{\frac{2}{3}}}{A} Z^2 - (4a_4 + (m_n - m_p)c^2)Z + C$$
$$= \frac{4a_4 + a_3 A^{\frac{2}{3}}}{A}\left(Z - \frac{A}{2} \times \frac{4a_4 + (m_n - m_p)c^2}{4a_4 + a_3 A^{\frac{2}{3}}}\right)^2 + \left(C - \frac{A}{4} \times \frac{(4a_4 + (m_n - m_p)c^2)^2}{4a_4 + a_3 A^{\frac{2}{3}}}\right)$$
$$= \frac{4a_4 + a_3 A^{\frac{2}{3}}}{A}(Z - Z_{\text{min}})^2 + \left(C - \frac{A}{4} \times \frac{(4a_4 + (m_n - m_p)c^2)^2}{4a_4 + a_3 A^{\frac{2}{3}}}\right), \tag{3.9}$$

where we have collected all the constant terms into C, namely,

$$C = Am_n c^2 - a_1 A + a_2 A^{\frac{2}{3}} + a_4 A \pm a_5 A^{-\frac{3}{4}}. \tag{3.10}$$

The mass formula depends quadratically on Z, and, when plotted against Z for fixed A, shows a minimum at $Z = Z_{\text{min}}$ as given in (3.2). For $A = 16$ and $A = 208$, the total number of nucleons is a multiple of 4. Consequently, we can have only an even–even structure (although an odd–odd nucleus has even A, it will not correspond to a multiple of 4), so that we need to consider only the negative sign in the last term of the mass formula in (3.1). For completeness, we plot in Figs. 3.1 and 3.2 the graphs for $A = 16$ and 208, respectively, for both odd–odd and even–even nuclei.