CHAPTER 10
Symmetries

Lecture Notes For
PHYS 415
Introduction to Nuclear and Particle Physics

To Accompany the Text
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World Scientific

Noether’s Theorem

- Emmy Noether proved that for every underlying symmetry, or invariance, of a system, there is a conserved quantity:
  - Space translations: momentum conservation
  - Rotations: angular momentum conservation
  - Time translations: energy conservation
  - Rotations in isospin space: isospin conservation
  - EM gauge invariance: charge conservation

Symmetries in Lagrangian Formalism

- Define the Lagrangian: *L* = *T* - *V*

$$L = L(q_i, \dot{q}_i) \quad \text{and} \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{with } i = 1,2,\dots,n$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

- If the Lagrangian is independent of some coordinate, then the corresponding *conjugate momentum* is conserved:

$$\frac{\partial L}{\partial q_m} = 0 \Rightarrow \frac{dp_m}{dt} = 0$$

Symmetries in Hamiltonian Formalism

- Define the Hamiltonian: $H = T + V$

$$H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t)$$

$$\frac{dq_i}{dt} = \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{with} \quad i = 1,2,\ldots,n$$

- Define the *Poisson bracket*:

$$\{F(q_i,p_i),G(q_i,p_i)\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right) = -\{G(q_i,p_i),F(q_i,p_i)\}$$

Canonical Poisson Brackets

- For the coordinates and momenta: $\{q_i, q_j\} = 0$

$$\{p_i, p_j\} = 0$$

$$\{q_i, p_j\} = -\{p_j, q_i\} = \delta_{ij}$$

- So we can rewrite Hamilton's equations: $\{q_i, H\} = \frac{\partial H}{\partial p_i} = \dot{q}_i$

$$\{p_i, H\} = -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

- For an observable which does not depend explicitly on time:

$$\frac{d\omega(q_i, p_i)}{dt} = \{\omega(q_i, p_i), H\}$$

Infinitesimal Translations

- Define an infinitesimal coordinate translation:

$$q_i \to q_i' = q_i + \varepsilon_i \Rightarrow \delta_\varepsilon q_i = q_i' - q_i = \varepsilon_i$$

$$p_i \to p_i' = p_i \qquad \Rightarrow \delta_\varepsilon p_i = p_i' - p_i = 0$$

- Define the function $g = \sum_j \varepsilon_j p_j$

  where p_j are generators of translation

- Then $\dfrac{\partial g}{\partial q_i} = 0$ and $\dfrac{\partial g}{\partial p_i} = \varepsilon_i$

$$\{q_i, g\} = \varepsilon_i = \delta_\varepsilon q_i$$

- And $\{p_i, g\} = 0 = \delta_\varepsilon p_i$

Infinitesimal Translations, cont'd.

- The translated variables obey the same canonical Poisson-bracket equations as the original ones:
$$\left\{q_i',q_j'\right\}=0=\left\{p_i',p_j'\right\} \text{ and } \left\{q_i',p_j'\right\}=\delta_{ij}$$
- Thus, these translations are termed **canonical transformations**.
- The Hamiltonian transforms as

$$\delta_\varepsilon H=\sum_i\left(\frac{\partial H}{\partial q_i}\delta_\varepsilon q_i+\frac{\partial H}{\partial p_i}\delta_\varepsilon p_i\right)=\sum_i\frac{\partial H}{\partial q_i}\varepsilon_i=\sum_i\left(\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i}\right)=\{H,g\}$$

Symmetry under Translations

- If the Hamiltonian is invariant under the translation:

$$\delta_\varepsilon H = \{H, g\} = 0 \Rightarrow H(q_i', p_i') = H(q_i, p_i)$$

the the transformed variables obey the same equations of motion as before:

$$\dot{q}_i' = \{q_i', H(q_j', p_j')\} = \{q_i, H(q_j, p_j)\}$$

$$\dot{p}_i' = \{p_i', H(q_j', p_j')\} = \{p_i, H(q_j, p_j)\}$$

- If the translations represent a symmetry of the system:

$$\frac{dg}{dt} = \{g, H\} = 0 \Rightarrow \frac{dp_i}{dt} = \{p_i, H\} = 0$$

Infinitesimal Rotations

- The change in coordinates for infinitesimal rotations (by angle ε) about the *z*-axis can be expressed via a matrix:

$$\delta_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \delta_\varepsilon \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

- Define the corresponding function g(ε):

$$g = \varepsilon\left(xp_y - yp_x\right) = \varepsilon\left(\vec{r} \times \vec{p}\right)_z = \varepsilon\ell_z \Rightarrow \begin{cases} \{x, g\} = \dfrac{\partial g}{\partial p_x} = -\varepsilon y = \delta_\varepsilon x \\ \{y, g\} = \dfrac{\partial g}{\partial p_y} = \varepsilon x = \delta_\varepsilon y \\ \{p_x, g\} = -\dfrac{\partial g}{\partial x} = -\varepsilon p_y = \delta_\varepsilon p_x \\ \{p_y, g\} = -\dfrac{\partial g}{\partial y} = \varepsilon p_x = \delta_\varepsilon p_y \end{cases}$$

Symmetry under Rotations

- The change in the Hamiltonian is

$$\delta_\varepsilon H = \sum_{i=1}^{2}\left(\frac{\partial H}{\partial q_i}\delta_\varepsilon q_i + \frac{\partial H}{\partial p_i}\delta_\varepsilon p_i\right) = \sum_{i=1}^{2}\left(\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i}\right) = \{H, g\}$$

$$\text{where } q_1 = x,\; q_2 = y,\; p_1 = p_x,\; p_2 = p_y$$

- If the Hamiltonian is invariant under the rotation:

$$\delta_\varepsilon H = \{H, g\} = 0 \Rightarrow \{g, H\} = \frac{dg}{dt} = \varepsilon\frac{d\ell_z}{dt} = 0$$

Symmetries in Quantum Mechanics

- Classical → Quantum:
  - Observable → Hermitian operator
  - Poisson bracket → Commutator
- Time evolution of an operator which does not depend explicitly on time is governed by Ehrenfest's theorem:

$$\frac{d}{dt}\langle Q\rangle = \frac{1}{i\hbar}\langle [Q,H]\rangle \quad \text{where} \quad \langle Q\rangle \equiv \langle \psi|Q|\psi\rangle$$

- The symmetry and corresponding conserved quantity are expressed as:

$$[Q,H]=0 \Rightarrow \frac{d}{dt}\langle Q\rangle = 0, \ \text{ if } Q \text{ has no explicit } t \text{ dependence}$$

Conserved Quantum Numbers

- If the operators *Q* and *H* commute, then we can define states which are simultaneous eigenfunctions of both.
- The energy eigenstates can then be labeled by quantum numbers corresponding to *Q*.
- For any process where the interaction Hamiltonian is invariant under a symmetry transformation, the corresponding quantum numbers are conserved.
- This explains why certain quantum numbers are conserved in some interactions but not others and provides clues to constructing the correct interaction Hamiltonian for various processes.

Infinitesimal Translations

- We perform a transformation on the state vectors:

$$x \rightarrow x' = x + \varepsilon \Rightarrow \psi(x) \rightarrow \psi(x-\varepsilon) = \psi(x) - \varepsilon \frac{d\psi(x)}{dx} + O\left(\varepsilon^2\right)$$

- Expectation values of the Hamiltonian can be shown to transform as (to first order in ε):

$$\langle H \rangle' = \langle H \rangle - \frac{i\varepsilon}{\hbar}\langle [H, p_x] \rangle \quad \text{where} \quad p_x = -i\hbar \frac{d}{dx}$$

- Comparing this to the classical case, we define the generating function g(ε) as:

$$\boxed{g = \varepsilon G = -\frac{i\varepsilon}{\hbar} p_x}$$

Symmetry under Translations

- The Hamiltonian will be invariant under translations of the *x*-coordinate if

$$\left[p_x, H \right] = 0$$

- In this case, Ehrenfest's theorem implies that the momentum is also conserved:

$$\frac{d}{dt}\langle p_x \rangle = 0$$

Continuous Symmetries

- Symmetries are either
  - Continuous: translations, rotations, …
  - Discrete: parity, time-reversal, … (i.e. "reflections")
- We can produce a finite translation by an infinite number of infinitesimal translations. Define the infinitesimal translation operator:

$$U_x(\varepsilon) = 1 - \frac{i\varepsilon}{\hbar} p_x$$

- For *N* successive translations:

$$U_x(N\varepsilon) = \left(1 - \frac{i\varepsilon}{\hbar} p_x\right)^N$$

Finite Translations

- Define a finite translation by amount $\alpha = N\varepsilon$ where $N \to \infty$ and $\varepsilon \to 0$, with $N\varepsilon$ finite:

$$U_x(\alpha) = \lim_{\substack{N \to \infty \\ \varepsilon \to 0 \\ N\varepsilon = \alpha}} \left(1 - \frac{i\varepsilon}{\hbar} p_x\right)^N = \lim_{\substack{N \to \infty \\ \varepsilon \to 0 \\ N\varepsilon = \alpha}} \left(1 - \frac{i\alpha}{N\hbar} p_x\right)^N = e^{-\frac{i}{\hbar}\alpha p_x}$$

- Therefore finite translations are obtained by exponentiating the generator for infinitesimal translations.

Abelian Groups

- Generators of translations correspond to a commutative (**Abelian**) group:

$$\left[p_i, p_j\right] = 0, \;\; i, j = x, y, \text{ or } z \Rightarrow$$

$$U_j(\alpha)U_k(\beta) = e^{-\frac{i}{\hbar}\alpha p_j} e^{-\frac{i}{\hbar}\beta p_k} = e^{-\frac{i}{\hbar}\beta p_k} e^{-\frac{i}{\hbar}\alpha p_j} = U_k(\beta)U_j(\alpha)$$

$$\text{and } \; U_x(\alpha)U_x(\beta) = e^{-\frac{i}{\hbar}\alpha p_x} e^{-\frac{i}{\hbar}\beta p_x} = U_x(\alpha+\beta) = U_x(\beta)U_x(\alpha)$$

- Translations are additive and the order is not relevant.

Rotations Do NOT Commute

- The generators of rotations are the angular momentum operators, obeying commutation relations:

$$\left[L_j, L_k\right] = \sum_{\ell} i\hbar \varepsilon_{jk\ell} L_\ell, \quad j,k,\ell = 1,2,3 = x,y,z$$

- Rotations therefore form a non-Abelian group: the order is important.
- Rotations in three dimensions correspond to the group *SO*(3) (real 3×3 matrices: <u>S</u>pecial <u>O</u>rthogonal, where "special" means determinant = +1).
- This group has a similar structure to *SU*(2) (complex, 2×2 matrices: <u>S</u>pecial <u>U</u>nitary) describing spin 1/2 states.

Spin 1/2

- Consider a two-level system represented by:

$$\begin{pmatrix} \psi_1(x) \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \psi_2(x) \end{pmatrix}$$

- A general rotation in the "internal" space (i.e. does not affect space-time coordinates) is given by:

$$\delta \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = -\sum_{j=1}^{3} i\varepsilon_j \frac{\sigma_j}{2} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \text{ where } \sigma_j \text{ are the Pauli matrices:}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } I_j = \frac{\sigma_j}{2}$$

SU(2) Symmetries

- If such a rotation corresponds to a symmetry of the system, then the two eigenstates of I_3 will be degenerate in energy.
  - Spin 1/2: in the absence of a magnetic field the spin-up and spin-down states are degenerate.
  - Isospin 1/2: the proton and neutron are degenerate if the Hamiltonian is invariant under isospin rotations. This is the case for the strong interaction.
    - In this case isospin is a conserved quantum number.

Isospin

- The proton and neutron states will transform under rotations in isospin space as:

$$|p'\rangle = \cos\frac{\theta}{2}|p\rangle - \sin\frac{\theta}{2}|n\rangle$$

$$|n'\rangle = \sin\frac{\theta}{2}|p\rangle + \cos\frac{\theta}{2}|n\rangle$$

- If we consider a two-nucleon system, then there are four possible states:

$$|\psi_1\rangle = |pp\rangle,\ |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right),\ |\psi_3\rangle = |nn\rangle$$

$$\text{and } |\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right)$$

The Two-Nucleon System

- Transforming these states, it can be shown that the first three transform into one another like the components of a vector, whereas the fourth is invariant.
  - $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ correspond to an isovector (I=1) triplet with I_3 = +1, 0 and -1 respectively.
  - $|\psi_4\rangle$ corresponds to an isoscalar (I=0) singlet with I_3 = 0.
- If the nucleon-nucleon strong interaction is isospin invariant, then the three I = 1 states are indistinguishable.
  - The two-nucleon system can be classified as either isovector or isoscalar.

Transition Rates for Δ decay

- The $\Delta(1232)$ is an $I = 3/2$ π-N resonance.
- The strong decay rates of the various Δ states should be equal:
  - $\Delta^{++} \rightarrow p\pi^{+}$ ($I_3 = 3/2$) Rate: 1 (arbitrary normalization)
  - $\Delta^{+} \rightarrow p\pi^{0}$ ($I_3 = 1/2$) Rate: x
    $\Delta^{+} \rightarrow n\pi^{+}$ Rate: $1-x$
  - $\Delta^{0} \rightarrow p\pi^{-}$ ($I_3 = -1/2$) Rate: y
    $\Delta^{0} \rightarrow n\pi^{0}$ Rate: $1-y$
  - $\Delta^{-} \rightarrow n\pi^{-}$ ($I_3 = -3/2$) Rate: 1
- Assuming that rates for p or n final states are the same and rates for π^{+}, π^{0}, π^{-} final states are the same, we can determine the rates of each decay relative to the rate for a given decay.
- The expected rates agree with data suggesting isospin is a symmetry of the strong interaction.

Local Symmetries

- Continuous symmetries can be:
  - **Global**: Same transformation at all space-time points. Results in conserved quantum numbers.
  - **Local**: Transformation depends on space-time coordinates. Requires explicit forces to maintain the symmetry.
- Global symmetry example:

$$\text{For TISE}: \quad H\psi(\vec{r}) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})$$

$$\text{If } \psi(\vec{r}) \text{ is a solution, so is}: \quad e^{i\alpha}\psi(\vec{r})$$

  - This symmetry preserves probability density.
  - Such a global phase transformation is associated with conservation of electric charge.

Gauge Fields

- Now consider a local phase transformation:

$$\psi(\vec{r}) \rightarrow e^{i\alpha(\vec{r})}\psi(\vec{r})$$

U(1) Abelian group

- The gradient introduces an inhomogeneous term:

$$\vec{\nabla}\left[e^{i\alpha(\vec{r})}\psi(\vec{r})\right] = e^{i\alpha(\vec{r})}\left[i\left(\vec{\nabla}\alpha(\vec{r})\right)\psi(\vec{r}) + \vec{\nabla}\psi(\vec{r})\right] \neq e^{i\alpha(\vec{r})}\vec{\nabla}\psi(\vec{r})$$

- The Schrödinger equation is not invariant under this transformation.
- We can retain the symmetry if we modify the gradient by introducing a vector potential with certain transformation properties:

$$\vec{\nabla} \rightarrow \vec{\nabla} - i\vec{A}(\vec{r})$$

$$\vec{A}(\vec{r}) \rightarrow \vec{A}(\vec{r}) + \vec{\nabla}\alpha(\vec{r})$$

Gauge Fields, cont'd.

- Then the combined transformation is:

$$\left(\vec{\nabla}-i\vec{A}(\vec{r})\right)\psi(\vec{r}) \to \left(\vec{\nabla}-i\vec{A}(\vec{r})-i\left(\vec{\nabla}\alpha(\vec{r})\right)\right)\left(e^{i\alpha(\vec{r})}\psi(\vec{r})\right)$$

$$= e^{i\alpha(\vec{r})}\left(\vec{\nabla}-i\vec{A}(\vec{r})\right)\psi(\vec{r})$$

- The local phase transformation is then a symmetry of the modified Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\left(\vec{\nabla}-i\vec{A}(\vec{r})\right)^2+V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})$$

- Invariance under a local phase transformation requires introduction of gauge fields.
- Fundamental forces arise from local invariances of physical theories, and the associated gauge fields generate the forces ⇒ **gauge theories**.