CHAPTER 11
Discrete Transformations

Lecture Notes For
PHYS 415
Introduction to Nuclear and Particle Physics

To Accompany the Text
Introduction to Nuclear and Particle Physics, 2nd Ed.
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Overview

- We will consider three discrete transformations
  - Parity: reflection through the origin
  - Time reversal: $t \rightarrow -t$
  - Charge conjugation: particles ⇔ antiparticles
- Both *P* and *C* are known to be violated in certain weak processes.
- The combined transformation *CP* is also violated in some systems.
- *CPT* Theorem:
  - No known interaction violates the combination *CPT*.
  - *CPT* invariance can be proven to be a consequence of certain fundamental assumptions (*CPT* Theorem).
  - *CP* violation + *CPT* invariance ⇒ *T* violation. However, no direct *T* violation has yet been observed.

Parity Transformation

- Parity corresponds to spatial inversion:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \xrightarrow{P} \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}$$

- This converts a right-handed system into a left-handed system or vice versa.
- No rotation or set of rotations can produce this transformation: the quantum numbers for parity and rotations are distinct.
- Note that parity is a discrete transformation whereas rotations, whether in real space, spin space or isospin space, are continuous transformations.

Vectors and Scalars Under Parity

- Under parity vectors transform as:

$$\vec{r} \rightarrow -\vec{r},$$

$$\vec{p} = m\dot{\vec{r}} \rightarrow -m\dot{\vec{r}} = -\vec{p}$$

- The magnitudes are unchanged:

$$r = \sqrt{\vec{r} \cdot \vec{r}} \rightarrow r$$

$$p = \sqrt{\vec{p} \cdot \vec{p}} \rightarrow p$$

Axial Vectors and Pseudoscalars

- The orbital angular momentum does not transform like an ordinary vector:

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{P} (-\vec{r}) \times (-\vec{p}) = \vec{r} \times \vec{p} = \vec{L}$$

- Vectors which transform this way (i.e. positive parity) are called pseudovectors or **axial vectors**.
- Certain scalars transform oppositely to normal scalars, such as the volume of a parallelopiped:

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \xrightarrow{P} \left(-\vec{a}\right) \cdot \left(-\vec{b} \times -\vec{c}\right) = -\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$$

- Such scalars, with negative parity, are called pseudoscalars.

Parity Quantum Numbers

- Applying the parity operator twice leaves the coordinate system unchanged:

$$\vec{r} \xrightarrow{P} -\vec{r} \xrightarrow{P} \vec{r}$$

- Therefore, the possible quantum numbers of the parity operator, *P*, are ±1:

$$P^2|\psi\rangle = +1|\psi\rangle = \lambda^2|\psi\rangle \Rightarrow \lambda = \pm 1$$

Eigenstates of Parity

- If the Hamiltonian is invariant under spatial inversion $[P,H] = 0$.
- In this case we can find common eigenstates of *H* and *P,* where the parity eigenvalues are ± 1.
- Under parity the wave function becomes:

$$\psi(\vec{r}) \xrightarrow{P} \psi(-\vec{r})$$

- Therefore the eigenstates of any Hamiltonian which is invariant under spatial inversion can be classified as either even or odd functions.
- Examples:
  - Square well potential.
  - Simple harmonic oscillator.
  - Central potential …

Parity and Central Potentials

- Using spherical coordinates, the parity transformation is:

$$r \xrightarrow{P} r$$

$$\theta \xrightarrow{P} \pi - \theta$$

$$\phi \xrightarrow{P} \pi + \phi$$

- The spherical harmonics transform as:

$$Y_{\ell m}(\theta,\phi) \xrightarrow{P} Y_{\ell m}(\pi-\theta,\pi+\phi) = (-1)^{\ell} Y_{\ell m}(\theta,\phi)$$

- Therefore the wave function transforms as:

$$\psi_{n\ell m}(\vec{r}) \xrightarrow{P} (-1)^{\ell} \psi_{n\ell m}(\vec{r})$$

Intrinsic Parity

- A quantum state can also have an *intrinsic parity* independent of its spatial transformation properties. Including this, the wave function transforms as:

$$\psi_{n\ell m}(\vec{r}) \xrightarrow{P} \eta_\psi (-1)^\ell \psi_{n\ell m}(\vec{r}) \quad \text{with} \quad \eta_\psi^2 = 1$$

- The total parity is then:

$$\eta_{\text{TOT}} = \eta_\psi (-1)^\ell$$

- Bosons: η_ψ same for particle and antiparticle.
- Fermions: η_ψ opposite for particle and antiparticle.
- Both Newton's laws and Maxwell's equations are invariant under parity.

Intrinsic Parity Assignments

- Absolute intrinsic parity for particles cannot be defined, since changing parity of all particles amounts to introducing an overall phase in every wave function.
- By convention: parity of proton, neutron and Λ hyperon are +1.
- Intrinsic parities of other particles can be established by considering parity conserving processes involving such particles.

Parity Conservation

- Consider the decay in the rest frame of a particle into two spinless particles: $$A \rightarrow B + C$$

- Angular momentum is conserved:

  $J_{\text{initial}} = J_{\text{final}} \equiv J \Rightarrow J = \ell$

  where ℓ = relative orbital angular momentum of B and C.

- If parity is conserved in the decay:

$$\eta_A = \eta_B \eta_C (-1)^{\ell} = \eta_B \eta_C (-1)^{J}$$

- If *A* is spinless also: $\eta_A = \eta_B \eta_C$

- So, for example $J^P = 0^+ \rightarrow 0^- + 0^-$ whereas $0^+ \not\rightarrow 0^+ + 0^-$

Parity of π^- Meson

- Consider the capture of a low-energy π^- meson on a deuteron: $\pi^- + d \rightarrow n + n$.
- Conservation of parity implies:

$$\eta_\pi \underbrace{\eta_d}_{=1} (-1)^{\ell_i} = \underbrace{\eta_n \eta_n}_{=1} (-1)^{\ell_f} \Rightarrow \eta_\pi = (-1)^{\ell_f + \ell_i}$$

- Since it is known that $\ell_i = 0$: $\eta_\pi = (-1)^{\ell_f}$

- We can determine ℓ_f from symmetry requirements …

Parity of π^- Meson, cont' d.

- The spin of the deuteron is 1 $\Rightarrow$ the final total angular momentum is 1:

3 possibilities
$$\begin{cases} \left|\psi_{nn}^{(1)}\right\rangle = \left|J=1, s=1, \ell_f = 0 \text{ or } 2\right\rangle \\ \left|\psi_{nn}^{(2)}\right\rangle = \left|J=1, s=1, \ell_f = 1\right\rangle \\ \left|\psi_{nn}^{(3)}\right\rangle = \left|J=1, s=0, \ell_f = 1\right\rangle \end{cases}$$

- The two-neutron state must be antisymmetric under interchange, since it consists of two identical fermions.
- This implies we have either a symmetric spin wave function coupled with an antisymmetric spatial wave function or vice versa.
- Only the second possibility is allowed $\Rightarrow$ $\eta_\pi = -1$.

Violation of Parity

- In the early 1950's the decays of two particles (called the τ and θ) which had essentially identical masses and lifetimes, presented a dilemma (the "τ - θ" puzzle):

$$\theta^{+} \rightarrow \pi^{+} + \pi^{0}$$

$$\tau^{+} \rightarrow \pi^{+} + \pi^{+} + \pi^{-}$$

- If parity is conserved in these decays, the τ and θ can be shown to have opposite parity (see next slide). This suggests that either
  - The τ and θ are different particles and the extreme similarity is just a coincidence, or
  - Parity is violated in these decays.

As Aside: Parity Arguments

- The τ and θ were found to have J = 0.
- For the two particle final state this implies ℓ_f = 0.
- For the three particle final state there are two orbital angular momenta to consider, but both were found to be zero.
- This implies that the intrinsic parities are:
  - $\eta_\theta = \eta_\pi \times \eta_\pi = (-1)^2 = +1$
  - $\eta_\tau = \eta_\pi \times \eta_\pi \times \eta_\pi = (-1)^3 = -1$
- The intrinsic parities of the τ and θ are opposite.

Parity is Violated in Weak Interactions

- Lee and Yang postulated parity is violated in weak interactions. The τ and θ are now indeed known to be the same particle: K^+.
- The conclusive proof of parity violation was provided in an experiment by Wu *et al*.

$$^{60}\mathrm{Co} \rightarrow {}^{60}\mathrm{Ni} + e^- + \bar{\nu}_e$$

  - The cobalt nuclei were polarized in a strong magnetic field.
  - The electrons were emitted preferentially in a direction opposite the field (opposite the spin of the nucleus).

Analysis of Wu Experiment

- Consider the spin ***s*** of ^{60}Co and the electron momentum ***p***:

$$\left\langle \cos\theta_e \right\rangle = \left\langle \frac{\vec{s}\cdot\vec{p}}{|\vec{s}||\vec{p}|} \right\rangle \xrightarrow{P} \left\langle \frac{\vec{s}\cdot(-\vec{p})}{|\vec{s}||\vec{p}|} \right\rangle = -\left\langle \cos\theta_e \right\rangle$$

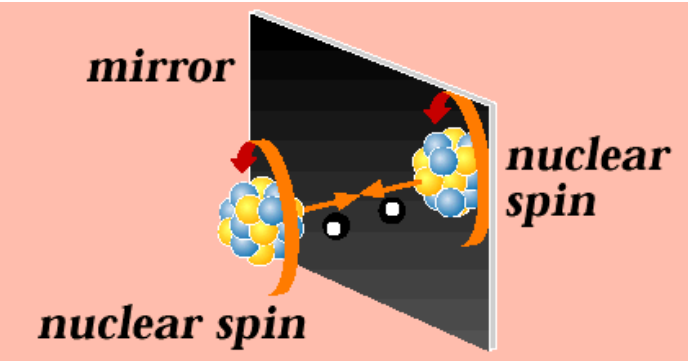
- If parity is conserved in the decay, the right and left handed coordinate systems are equivalent and we must have:

$$\left\langle \cos\theta_e \right\rangle \propto \left\langle \vec{s}\cdot\vec{p} \right\rangle = 0$$

- The preferential direction of electron emission implies a negative value, instead of zero. Parity is violated in this weak decay.

Physics distinguishes Left from Right

- The observed nonzero (negative) value confirms that the two coordinate systems are distinguishable.
- For amusement, consider how you might use this fact to communicate to an alien, without pictures, that you are right-handed.



From http://www.lbl.gov/abc/wallchart/chapters/05/graphics/Image2.gif

Time Reversal

- Time reversal corresponds to changing *t* to -*t* :

$$t \xrightarrow{T} -t$$

$$\vec{r} \xrightarrow{T} \vec{r}$$

$$\vec{p} = m\dot{\vec{r}} \xrightarrow{T} -m\dot{\vec{r}} = -\vec{p}$$

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{T} \vec{r} \times \left(-\vec{p}\right) = -\vec{L}$$

- Newton's 2nd Law is second order in the time derivative and so is invariant under *T*.
- Maxwell's equations are also invariant under *T*.
- Statistical mechanics implies entropy increases. This defines a unique direction for the flow of time for macroscopic systems. However, microscopic systems appear to respect *T* invariance in almost all cases.

Time Reversal in Quantum Mechanics

- Consider the time dependent S.E.:

$$i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = H\psi(\vec{r},t)$$

- The equation is first order in time and so would appear not to be invariant under *T*.
- However consider the complex conjugate equation (for *H* Hermitian):

$$-i\hbar \frac{\partial \psi^*(\vec{r},t)}{\partial t} = H\psi^*(\vec{r},t) \xrightarrow{T} i\hbar \frac{\partial \psi^*(\vec{r},-t)}{\partial t} = H\psi^*(\vec{r},-t)$$

- If the wave function transforms as $\psi(\vec{r},t) \xrightarrow{T} \psi^*(\vec{r},-t)$ then both ψ and its time reversed solution obey the same equation.

Detailed Balance

- Since time dependent wave functions are complex, they are not eigenfunctions of *T*.
- Consequently, there is no quantum number which can be associated with *T* invariance.
- In QM, *T* invariance implies transition amplitudes are the same for the forward and reversed processes:

$$\left| M_{i \to f} \right| = \left| M_{f \to i} \right|$$

- This *principle of detailed balance* has been verified for many processes. Note: the transition rates can be different however, since the density of final states can be quite different for the forward and reverse processes.

Neutron Electric Dipole Moment?

- The best evidence for *T* invariance comes from searches for a nonzero neutron EDM.
- The neutron, though neutral, has an extended charge distribution which gives rise to a magnetic moment.
- If the centers of the positive and negative charge distributions do not coincide, the neutron would also have an EDM.
- A naïve estimate of the size of the EDM:

$$\mu_{el} \le ed \approx e \times 10^{-13}\ \text{cm} \ \approx 10^{-13}\ e\text{-cm}$$

- The only possible axis is the neutron spin; a non-vanishing EDM would therefore have to point along the spin direction. The most sensitive search for this effect gives an upper limit:

$$\mu_{el} \le 10^{-25}\ e\text{-cm !}$$

Neutron EDM would Violate *T* Invariance

- Consider the component of EDM along the spin:

$$\vec{\mu}_{el} \cdot \vec{s} \xrightarrow{T} \vec{\mu}_{el} \cdot (-\vec{s}) = -\vec{\mu}_{el} \cdot \vec{s} \Rightarrow \langle \vec{\mu}_{el} \cdot \vec{s} \rangle \xrightarrow{T} -\langle \vec{\mu}_{el} \cdot \vec{s} \rangle$$

- This expectation value must vanish if electromagnetic interactions are *T* invariant.
- However, note that under parity:

$$\langle \vec{\mu}_{el} \cdot \vec{s} \rangle \xrightarrow{P} \langle (-\vec{\mu}_{el}) \cdot \vec{s} \rangle = -\langle \vec{\mu}_{el} \cdot \vec{s} \rangle$$

- So a nonzero expectation value can arise from parity violation. Other experiments indicate that electromagnetic interactions are *P* invariant. So a non-zero neutron EDM could arise from an interplay of EM and weak interactions.

Charge Conjugation

- Unlike *P* and *T* which are discrete space-time symmetry transformations, charge conjugation, *C*, operates on the internal state of a system:

$$Q \xrightarrow{C} -Q \quad \Longrightarrow \quad \begin{cases} \vec{E} \xrightarrow{C} -\vec{E} \\ \vec{B} \xrightarrow{C} -\vec{B} \end{cases}$$

- Maxwell's equations are invariant under *C*.
- Charge conjugation inverts all internal quantum numbers of states, changing particles into antiparticles and vice versa.

Eigenstates of *C*

- Denoting the internal quantum numbers collectively as *Q*: $$\left|\psi(Q,\vec{r},t)\right\rangle \xrightarrow{C} \left|\psi(-Q,\vec{r},t)\right\rangle$$
- Therefore, neutral particles can be eigenstates of *C*: γ (photon), π^0 , …
- However, as particles carry quantum numbers other than charge, not all neutral particles are eigenstates of *C*:

$$\left|n\right\rangle \xrightarrow{C} \left|\bar{n}\right\rangle$$

$$\left|\pi^- p\right\rangle \xrightarrow{C} \left|\pi^+ \bar{p}\right\rangle$$

$$\left|K^0\right\rangle \xrightarrow{C} \left|\overline{K^0}\right\rangle$$

Charge Parity

- As for parity, two successive C operations leaves the system unchanged $\Rightarrow$ *charge parity* = ± 1.
- Since the photon is the carrier of the EM field: $\eta_C(\gamma) = -1$.
- If C is a symmetry of the theory then $[C,H] = 0$ and the charge parity for any process is conserved.
- Charge parity is conserved in EM processes since Maxwell's equations are C invariant.

Decay of π^0

- Consider the two photon decay of the π^0 :

$$\pi^0 \rightarrow \gamma + \gamma$$

- If charge parity is conserved:

$$\eta_C\left(\pi^0\right) = \eta_C(\gamma)\eta_C(\gamma) = (-1)^2 = +1$$

- Therefore *C* invariance implies the π^0 cannot decay to an odd number of photons:

$$\frac{\pi^0 \rightarrow 3\gamma}{\pi^0 \rightarrow 2\gamma} < 10^{-8}$$

Weak Interactions Violate Charge Conjugation

- Charge conjugation does not affect space-time properties and therefore, handedness:

$$|\nu_L\rangle \xrightarrow{C} |\overline{\nu_L}\rangle \text{ and } |\overline{\nu_R}\rangle \xrightarrow{C} |\nu_R\rangle$$

- Since there is no evidence for right handed neutrinos or left handed antineutrinos, the charge conjugate process of β-decay cannot occur $\Rightarrow$ weak interactions violate *C* invariance.
- However, under the combined operation of *CP* :

$$|\nu_L\rangle \xrightarrow{P} |\nu_R\rangle \xrightarrow{C} |\overline{\nu_R}\rangle$$

$$|\overline{\nu_R}\rangle \xrightarrow{P} |\overline{\nu_L}\rangle \xrightarrow{C} |\nu_L\rangle$$

- *CP* takes a physical state to another physical state and is a symmetry of almost all processes. *CP* violation, though small, has interesting possible implications for the matter-antimatter asymmetry of the universe.

CPT Theorem

- Though *P*, *T* and *C* appear to be violated in some processes, Lüders, Pauli and Schwinger showed that the combined operation of *CPT* is a symmetry of essentially any theory which respects Lorentz invariance.
- This is known as the ***CPT* theorem** and it is consistent with all observations to date.

Consequences of *CPT* Theorem

- The *CPT* theorem leads to various conclusions:
  - Particles with integer spin obey Bose-Einstein statistics and particles with half-integer spin obey Fermi-Dirac statistics.
  - Particles and their antiparticles have the same masses and same total lifetimes.
  - All the internal quantum numbers are opposite those of their partner particles.