CHAPTER 3

Nuclear Models

Lecture Notes For
PHYS 415
Introduction to Nuclear and Particle Physics

To Accompany the Text
Introduction to Nuclear and Particle Physics, 2nd Ed.
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World Scientific

General Remarks

- The nuclear force has proved elusive.
  - Scattering experiments gave many clues, but, at the time, no fundamental theory of the nuclear force existed.
  - Even when the fundamental theory was developed (QCD), due to the strong coupling nature of the force, first principle calculations were impossible.
  - Lately, various techniques have been developed, such as lattice QCD, but, currently, they too are limited in what they are able to predict.
- Nuclear models were developed as a result.
  - Phenomenological basis.
  - Limited range of validity.

Liquid Drop Model

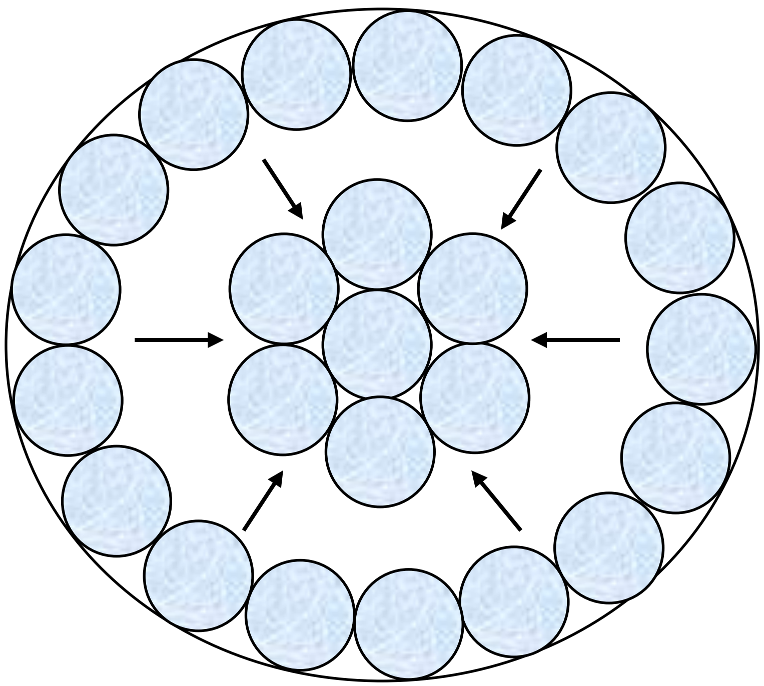
- Nuclear densities are almost independent of nucleon number:

$$R = r_0 A^{1/3} \Rightarrow \rho = \frac{M}{V} \approx \frac{m_N A}{\frac{4}{3}\pi R^3} = \frac{m_N A}{\frac{4}{3}\pi r_0{}^3 A} = \frac{m_N}{\frac{4}{3}\pi r_0{}^3} = \text{ constant}$$

- This suggests the nucleus as an incompressible fluid: **liquid drop**.
- Adding more nucleons, increases the size, but not the density.
  - Only nearest neighbor interactions are important.
  - Nuclear force *saturates*.
  - Consistent with B/A ≈ constant.

Surface Tension

- Nucleons at the surface feel forces only from interior nucleons.
  - The force is unbalanced at the surface.
  - Since the force is attractive, there is a net inward attraction of the surface: *surface tension*.
  - Surface nucleons are less tightly bound.



Binding Energy, Including Surface Effect

- Less tightly bound surface nucleons imply a positive correction to the binding energy:

$$\text{B.E.} = -a_1 A + a_2 A^{2/3}$$

Volume energy

Surface correction

- Surface effect is more important for lighter nuclei.
  - Higher surface-to-volume ratio
  - Therefore, light nuclei are less tightly bound

Coulomb Forces

- The coulomb repulsion among protons decreases the binding energy per nucleon:

$$\text{B.E.} = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}}$$

$$\text{since } R \propto A^{1/3}$$

Bethe-Weizsäcker Semi-Empirical Mass Formula

- We still need to account for other observations:
  - Light nuclei with $N = Z$ are more abundant (stable)
  - Even-even nuclei are more abundant (stable).

$$\text{B.E.} = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}} + a_4 \frac{(N-Z)^2}{A} \pm a_5 A^{-3/4}$$

$$\Rightarrow M(A,Z) = (A-Z)m_n + Zm_p$$

$$-\frac{a_1}{c^2} A + \frac{a_2}{c^2} A^{2/3} + \frac{a_3}{c^2} \frac{Z^2}{A^{1/3}} + \frac{a_4}{c^2} \frac{(A-2Z)^2}{A} \pm \frac{a_5}{c^2} A^{-3/4}$$

where the last term is

- \+ For odd-odd nuclei
- \- For even-even nuclei
- 0 For odd A nuclei

and, from an empirical fit

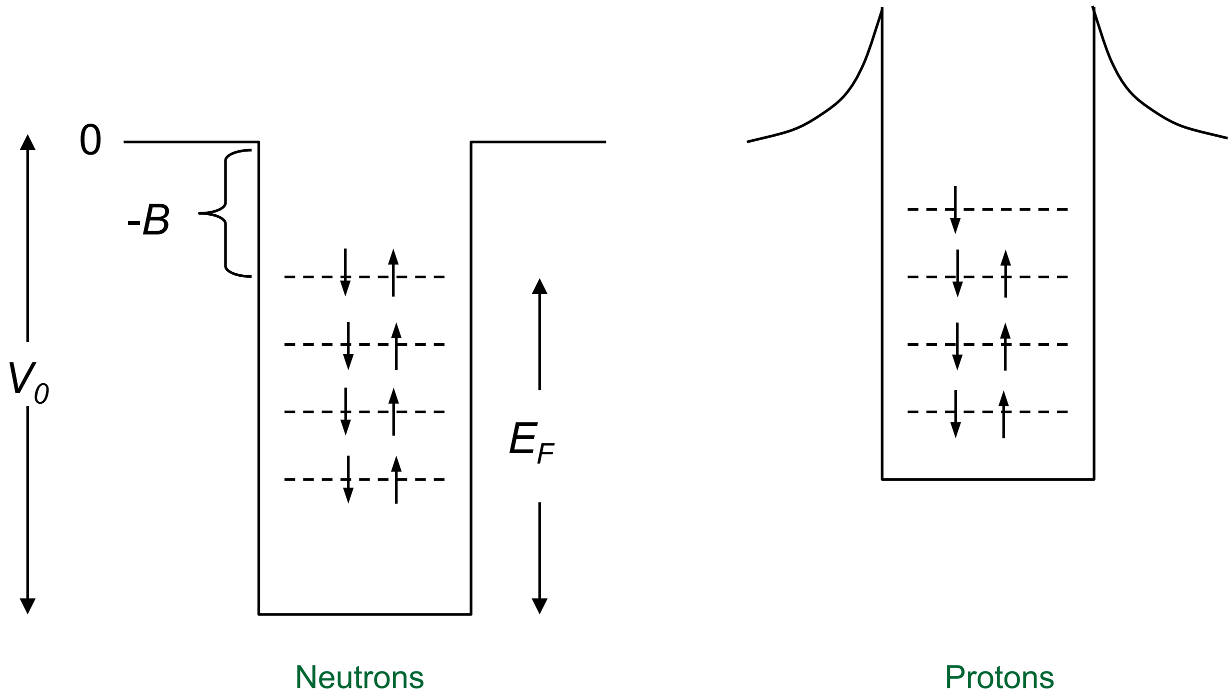
$a_1 \approx 15.6$ MeV $a_2 \approx 16.8$ MeV $a_3 \approx 0.72$ MeV
$a_4 \approx 23.3$ MeV $a_5 \approx 34$ MeV

Fermi-Gas Model

- Quantum-mechanical description where nucleons are considered a gas of fermions confined to a spherically symmetric potential well.
- Boundary conditions imply energy levels are discrete.
- Depth and range of well are fit to data.
- Only two identical fermions (of opposite spin projection) can occupy the same energy level.
  - Protons and neutrons may be considered distinguishable and so each level can contain four nucleons.
  - Protons experience the Coulomb force and so the potentials are slightly different for protons vs. neutrons.

Fermi Levels and Fermi Energy

- Protons occupy shallower well since heavy nuclei are neutron rich. Otherwise, neutrons in upper levels could undergo β^- decay to become lower energy protons.



Fermi Energy

- Nonrelativistically, $E_F = p_F^2 / 2m$
- Phase space volume = $\int d^3r \, d^3p$:

$$V_{TOT} = V \times V_F = \frac{4\pi}{3} r_0^{\,3} A \times \frac{4\pi}{3} p_F^{\,3} = \left(\frac{4\pi}{3}\right)^2 (r_0 p_F)^3 A$$

- Heisenberg: $\Delta x \Delta p_x \geq \frac{\hbar}{2} \Rightarrow V_{\text{state}} = (2\pi\hbar)^3 = h^3$
- Number of fermions up to E_F:

2 spin states

$$n_F = 2\frac{V_{TOT}}{V_{\text{state}}} = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3$$

Fermi Energy, cont'd.

- Assuming $N = Z = A/2$:

$$N = Z = \frac{A}{2} = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar} \right)^3 \Rightarrow p_F = \frac{\hbar}{r_0} \left(\frac{9\pi}{8} \right)^{1/3}, \text{ indep. of } A$$

$$\Rightarrow E_F = \frac{1}{2m} \left(\frac{\hbar}{r_0} \right)^2 \left(\frac{9\pi}{8} \right)^{2/3} \approx \frac{2.32}{2mc^2} \left(\frac{\hbar c}{r_0} \right)^2 \approx 33 \text{ MeV}$$

- Taking B.E. of last nucleon as ~8 MeV:

$$V_0 = E_F + B \approx 40 \text{ MeV}$$

Shell Model

- Nucleons within nuclei, like electrons within atoms, can be well described by a **shell model**.
- Four nucleons (two protons and two neutrons) can occupy each orbital.
- For atoms, this is natural, since the nucleus provides a central force. For nuclei there is no central force!
- The Pauli principle still forbids nucleons from occupying the same states and this gives rise to an effective **mean field potential** in which the nucleons move.

Review of Atomic Shell Model

- Electrons occupy states defined by four quantum numbers: n, ℓ, m_ℓ, m_s
  - $n = 1, 2, 3, \ldots$ = principal quantum #
  - $\ell = 0, 1, 2, \ldots, n-1$ = orbital quantum #
  - $m_\ell = -\ell, -\ell+1, \ldots, 0, 1, \ell-1, \ell$ = magnetic quantum #
  - $m_s = \pm 1/2$
- Considering only the nuclear Coulomb potential, which is rotationally symmetric, all states of a given n are *degenerate*: # of states = $2n^2$.

Spin-Orbit Interaction

- In atoms, the spin of an electron couples to the nuclear orbital motion (as seen in the electron rest frame), giving rise to small splittings: *fine structure*.
- In nuclei, there is also a spin-orbit interaction and this affects nuclear structure significantly.

Closed-Shell Atoms

- In atoms there is a strong pairing effect which effectively minimizes the angular momentum of the ground state.
  - $\sum m_s = 0$ and $\sum m_\ell = 0$
  - $\vec{L} = 0 = \vec{S}$ and $\vec{J} = \vec{L} + \vec{S} = 0$
- Closed shell elements are very stable.
  - Chemically inert
  - Large ionization energies

Magic Numbers

- Atomic shell closures occur at certain *magic numbers*: $Z = 2, 10, 18, 36, 54$
- Nuclear shell closures also occur at well-defined magic numbers:
  - $N = 2, 8, 20, 28, 50, 82, 126$
  - $Z = 2, 8, 20, 28, 50, 82$
- Nuclei where both protons and neutrons have closed shell are called **doubly magic** and have even greater stability.

Other Evidence for Nuclear Shells

- Closed neutron shell nuclei have more *isotones* and closed proton shell nuclei have more *isotopes* than neighboring nuclei.
- Neutron capture cross sections are relatively small for closed-shell nuclei.
- Proton and neutron knockout experiments can probe nucleons in individual orbits.
  - Can measure binding energies this way.
  - Can deduce the momentum distribution (in the context of a reaction model). The Fourier Transform of this gives the shape of the orbital.

Schrödinger Equation and Solutions

- For a central potential:

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r})$$

- Since the potential is spherically symmetric:

$$\left[H,\vec{L}^2\right] = 0 \Rightarrow \text{ sol'ns. are simultaneous eigenstates of } H \text{ and } \vec{L}^2$$

- Solutions can be written as:

$$\psi_{n\ell m_\ell}(\vec{r}) = \frac{u_{n\ell}(r)}{r} Y_{\ell m_\ell}(\theta,\phi)$$

Radial and Angular Equations

- Angular equations:

$$\vec{L}^2 Y_{\ell m_\ell}(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_{\ell m_\ell}(\theta,\phi)$$

$$L_z Y_{\ell m_\ell}(\theta,\phi) = m_\ell \hbar Y_{\ell m_\ell}(\theta,\phi)$$

- Radial equation:

$$\left[\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E_{n\ell} - V(r) - \underbrace{\frac{\hbar^2 \ell(\ell+1)}{2mr^2}}_{\text{Centrifugal barrier}}\right)\right] u_{n\ell}(r) = 0$$

Centrifugal barrier

Boundary conditions: $u_{n\ell}(r) \xrightarrow{r\to 0,\infty} 0$

Parity

- The parity operation is defined by: $P\psi(\vec{r}\,) = \psi(-\vec{r}\,)$
- This is equivalent to: $\begin{cases} r \to r \\ \theta \to \pi - \theta \\ \phi \to \pi + \phi \end{cases}$
- For spherically symmetric potentials, states have definite parity, related to the value of ℓ:

  parity = $(-1)^{\ell}$

  $\Rightarrow$ parity = + for ℓ even

  \- for ℓ odd

Infinite Square Well

- We can gain some intuition by solving this equation for simple potentials, even though they might not be entirely reasonable.
- For the ∞ square well:

$$V(r) = \begin{cases} \infty & r \geq R \\ 0 & \text{otherwise} \end{cases}$$

- We require that the solutions be regular at the origin, so we get the *spherical Bessel functions*:

$$u_{n\ell}(r) = j_\ell(k_{n\ell} r) \ \text{ where } \ k_{n\ell} = \sqrt{\frac{2mE_{n\ell}}{\hbar^2}}$$

Energies are Quantized

- Since the potential is infinite at $r = R$, we must have:

$$u_{n\ell}(R) = j_\ell(k_{n\ell}R) = 0, \quad \ell = 0,\ 1,\ 2,\ 3,\ \cdots$$
$$\text{and } \ n = 1,\ 2,\ 3,\ \cdots \quad \text{for any } \ell$$

- The energies are quantized and depend on *n* **and** ℓ. Each energy level can contain $2(2\ell+1)$ protons or neutrons.
- For *n* = 1, we get shell closures at:

  2, 2+6=**8**, 8+10=**18**, 18+14=**32**, 32+18=**50**, …

  (observed: 2, 8, 20, 28, 50, 82, …)
- Some, but not all, magic numbers are reproduced.

Harmonic Oscillator

- The three dimensional harmonic oscillator:

$$V(r) = \frac{1}{2} m\omega^2 r^2$$

- Solutions are related to the *associated Laguerre polynomials* with additional Gaussian factor (dominant dependence for large *r*).
- We also get bound states with discrete energies:

$$E_{n\ell} = \hbar\omega\left(2n + \ell - \frac{1}{2}\right)$$
and $\ell = 0,\ 1,\ 2,\ \ldots,$
for any $n = 1,\ 2,\ \ldots$

$$\xrightarrow{\Lambda \equiv 2n + \ell - 2}$$

$$E_{n\ell} = \hbar\omega\left(\Lambda + \frac{3}{2}\right)$$
$\Lambda = 0,\ 1,\ 2,\ \ldots$

Shell Closures

- Levels with different (n,ℓ), leading to the same Λ, will be degenerate. The degeneracy is:

  $n_\Lambda = (\Lambda+1)(\Lambda+2)$
- Therefore, shell closures occur for proton or neutron numbers of 2, 8, 20, 40, 70, …

  (observed: 2, 8, 20, 28, 50, 82, …)
- Again, some, but not all, magic numbers are reproduced.
- So introduce …

Spin-Orbit Potential

- Maria Goeppert Mayer and Hans Jensen (1949) suggested adding a spin-orbit interaction, in analogy with atoms:

$$V_{\mathrm{TOT}} = V(r) - f(r)\vec{L} \cdot \vec{S}$$

- Similar to atoms, except
  - Presence of $f(r)$ function
  - $j = \ell + 1/2$ state has lower energy than $j = \ell - 1/2$

Splitting due to Spin-Orbit Interaction

- We can write $$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$
$$\Rightarrow \Delta = \Delta E_{n\ell}\left(j = \ell - \frac{1}{2}\right) - \Delta E_{n\ell}\left(j = \ell + \frac{1}{2}\right)$$
$$= \hbar^2\left(\ell + \frac{1}{2}\right)\int d^3r \left|\psi_{n\ell}(\vec{r})\right|^2 f(r)$$
- Splitting is larger for higher ℓ values, allowing level crossing.
- For appropriate $f(r)$, we can reproduce all the magic numbers.

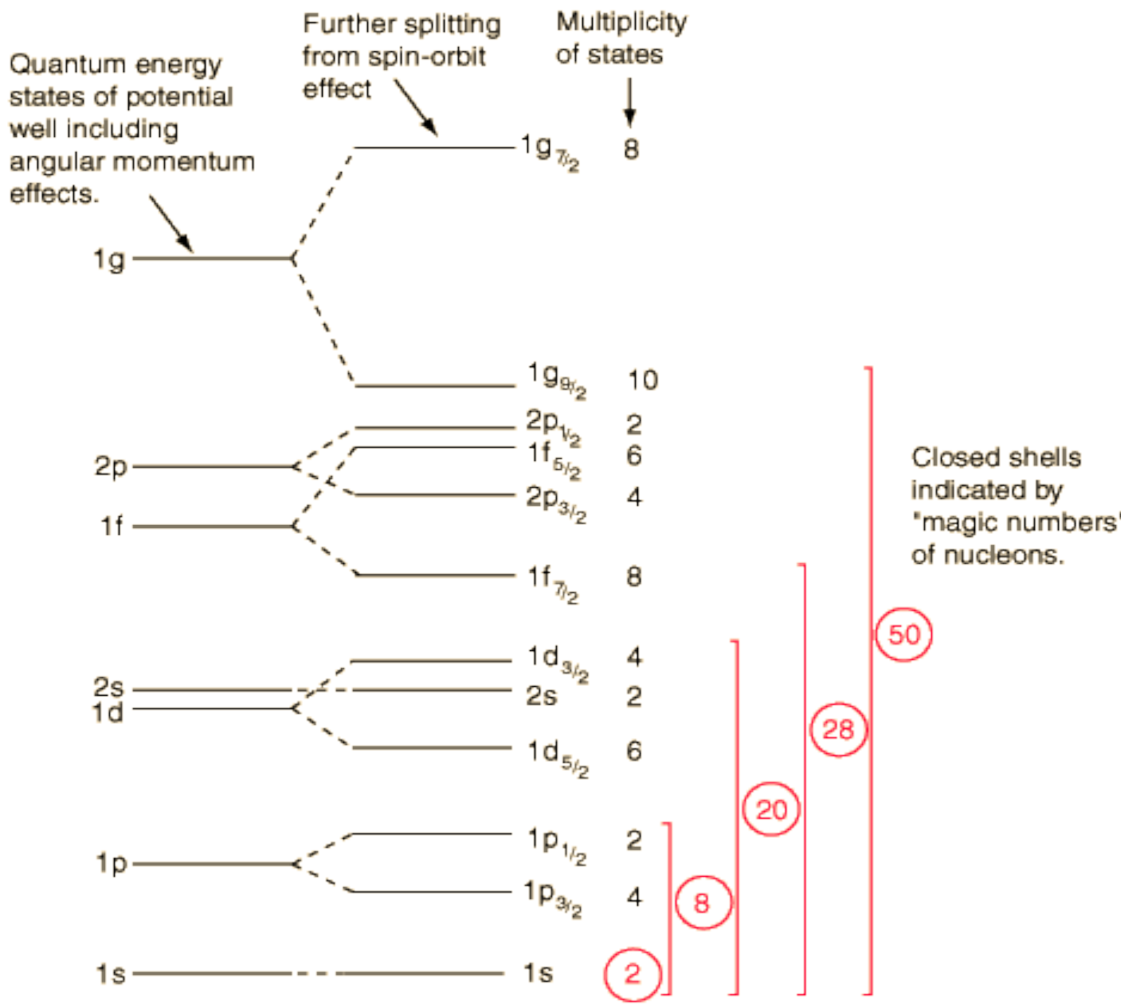
Spectroscopic Notation

- As for atoms, we use the spectroscopic notation to label states:

$$nL_j$$

- The L value is labeled by *S*, *P*, *D*, *F*, *G*, …, for $\ell = 0, 1, 2, 3, 4, \ldots$
- The multiplicity is $2j+1$

Level Scheme with Spin-Orbit Term



From: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/shell.html

Predictions of the Shell Model

- Spin-parity assignments of ground states of many odd-*A* nuclei predicted.
  - Neutrons and protons pair up with opposite spins, so that the spin of the last nucleon determines the spin of the nucleus.
  - Even-even nuclei, consequently, have zero spin, in agreement with observations.
- Allowing pairing between all valence nucleons can fix up agreement with remaining odd-*A* nuclei.

Example of Spin-Parity Prediction

- Consider $^{13}C^{6}$.
- Six protons and six neutrons are completely paired off.
- The last neutron will be in the shell: $1P_{1/2}$.
- So we expect $(1/2)^{-}$ (i.e. $j = 1/2$ and negative parity, since ℓ is odd).
- This is in agreement with observation.

Magnetic Moments of Nuclei

- We expect the nuclear magnetic moment to be determined by the moment of the unpaired nucleon(s).
- Each unpaired nucleon contributes:
  - Intrinsic (spin): $\mu_p = 2.79\ \mu_N$ and $\mu_n = -1.91\ \mu_N$
  - Orbital: protons only; neutrons are uncharged.
- Example: deuteron, assuming proton and neutron are in $(1S)_{1/2}$ states:
  - Prediction: $\mu_d = 2.79\ \mu_N - 1.91\ \mu_N = 0.88\ \mu_N$
  - Observation: $0.86\ \mu_N$
- Example: ${}^3He^2$. Here, the magnetic moment is expected to be due to the unpaired neutron and this is roughly correct. This fact, has been exploited to use ${}^3He^2$ targets to deduce neutron properties.

Collective Model

- The single-particle shell model does not describe certain features of heavy nuclei, namely magnetic dipole moments and electric quadrupole moments.
  - Electric quadrupole moments arise from non-spherical charge distributions, which cannot be explained by a purely central potential.
- Including many-body physics where nucleons interact with one another can remedy the problem.
- Historically, various collective models were introduced which provided simpler, intuitive descriptions of heavy nuclei.

Deformed Rotators and Vibrators

- Aage Bohr, Ben Mottelson and James Rainwater proposed a collective model to explain observed moments.
- Tried to reconcile liquid drop and shell model.
  - Hard core of nucleons in filled shell-model states.
  - Surface motion (rotation) of valence nucleons.
  - Latter gives rise to nonspherical shape and rotational and vibrational energy spectra.

Ellipsoidal Nucleus

- Define surface of nucleus as:

$$ax^2 + by^2 + \frac{z^2}{ab} = R^2$$

- Mean potential given by:

$$V(x,y,z) = \begin{cases} 0 & \text{for } ax^2 + by^2 + \frac{z^2}{ab} \leq R^2 \\ \infty & \text{otherwise} \end{cases}$$

- Deformations into an ellipsoidal shape can also be induced through particle bombardment of heavy nuclei. This will be useful for discussing nuclear fission later.

Rotational and Vibrational Levels

- For rotations, define the Hamiltonian:

$$H = \frac{\vec{L}^2}{2I}, \quad \text{with eigenvalues } = \frac{\ell(\ell+1)}{2I}\hbar^2$$

- For rotations perpendicular to symmetry axis, only expect even ℓ.
- Photon quadrupole transitions ($\Delta\ell = 2$) have been observed, corresponding to transitions between rotational levels.

Superdeformed Nuclei

- Heavy ion collision experiments have produced superdeformed nuclei with very large angular momentum quantum numbers.
- These deformed nuclei reach more spherical shapes by emitting a series of quadrupole γ-rays, each of order 50 keV.
- The photon energies remain essentially fixed. This is in conflict with the collective model, since the decreasing moment of inertia during "spin-down" should give rise to nonuniformly spaced photon energies.
- Emissions from different nuclei are also nearly identical. This too is in conflict with current models since effects of nucleon pairing should produce varying level spacings.