415 Final Exam (40 points)   May 8 2014, 3:45 ÷ 6:45. p.m

Problem 1. A photon hits the proton at rest and turns into $\pi^+\pi^-$ pair so in the final state we have proton, π^+, and π^-. What is the minimal photon energy required for this process? *Hint*: consider the problem in the c.m. frame.

Solution

$$s = (p+k)^2 = 2mE + m^2 \;\Rightarrow\; \sqrt{s} \;=\; \sqrt{2mE+m^2}$$

$\sqrt{s}$ should be bigger than $m + 2m_\pi = 1220$ MeV so

$$E_{\min} \;=\; \frac{(1220)^2 - (940)^2}{2\times 940}\text{ MeV} \;\simeq\; 321\text{ MeV}$$

Problem 2.

A beam of muons with kinetic energy 1 MeV travels through free space. What percentage of muons survives after traveling 500m?

Solution

Since the kinetic energy is much less than the rest energy the muon is non-relativistic so its velocity is

$$v \;=\; c\sqrt{\frac{2E}{m_\mu c^2}} \;=\; 4.1\times 10^7\frac{m}{s}$$

It will travel 500m in a time

$$t \;=\; \frac{500}{4.1}\times 10^{-7}s \;=\; 12.1\times 10^{-6}s$$

The half-time of muon is $t_{1/2} \;=\; \tau\ln 2$ where $\tau = 2.2\times 10^{-6}s$ so

$$\frac{N}{N_0} \;=\; e^{-\frac{t}{\tau}} \;=\; e^{-\frac{121}{22}} \;=\; 0.004 \;=\; 0.4\%$$

Problem 3. How the answer to Problem 3 would change for muons with the kinetic energy 1 GeV?

Solution

The total energy is now 1106 Mev so the muon is relativistic with

$$\gamma \;=\; \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \;=\; \frac{1106}{106} \;\simeq\; 10.43$$

The speed is approximately speed of light so the time of travel (in the lab frame) is

$$t_{\mathrm{lab}} \;=\; \frac{500m}{c} \;=\; 16.65\times 10^{-7}s$$

In the rest frame of the muon the elapsed time is

$$t_{\mathrm{rest}} \;=\; \frac{t_{\mathrm{lab}}}{\gamma} \;\simeq\; 1.6\times 10^{-7}$$

so

$$\frac{N}{N_0} \;=\; e^{-\frac{t_{\mathrm{rest}}}{\tau}} \;=\; e^{-\frac{1.6}{22}} \;=\; 0.93 \;=\; 93\%$$

Problem 4.

What is the ratio of expected ranges of propagation of a proton to that of a deuteron in aluminum? Assume that the initial velocities of proton and deuteron are equal and non-relativistic.

Solution

The stopping power for a non-relativistic particle is given by Eq. (6.3)

$$S(T) \;=\; \frac{4\pi Q^2 e^2 nZ}{m_e v^2}\ln\frac{m_e v^2}{\bar{I}}$$

and the range by (6.4)

$$R \;=\; \int_0^{T_0}\frac{dT}{S(T)} \;=\; m\int_0^{v_0} dv\frac{m_e v^3}{4\pi Q^2 e^2 nZ\ln\frac{m_e v^2}{\bar{I}}}$$

Since charges and initial velocities of proton and deuteron are equal,

$$\frac{R_d}{R_p} \;=\; \frac{m_d\int_0^{v_0} dv\frac{m_e v^3}{4\pi Q^2 e^2 nZ\ln\frac{m_e v^2}{\bar{I}}}}{m_p\int_0^{v_0} dv\frac{m_e v^3}{4\pi Q^2 e^2 nZ\ln\frac{m_e v^2}{\bar{I}}}} \;=\; \frac{m_d}{m_p} \;\simeq\; 2$$

Problem 5.

Determine if each of the following decays is allowed. If it is allowed determine the interaction by which it proceeds. If it is not allowed determine which conservation law(s) is/are violated.

$$\pi^+ \;\to\; \pi^0 + e^+ + \nu^e$$

$$K^- \;\to\; e^- + \nu^e$$

$$\mu^- \;\to\; \pi^- + \nu^\mu$$

$$\phi^- \;\to\; K^+ + K^-$$

Solution

$$\begin{aligned}
\pi^+ &\rightarrow \pi^0 + e^+ + \nu^e \ : \ \text{allowed, weak} \\
K^- &\rightarrow e^- + \nu^e \ : \ \text{violates lepton number} \\
\mu^- &\rightarrow \pi^- + \nu^\mu \ : \ \text{violates energy conservation} \\
\phi &\rightarrow K^+ + K^- \ : \ \text{allowed, strong}
\end{aligned}$$

Problem 6.

Determine if each of the following decays is allowed. If it is allowed determine the interaction by which it proceeds. If it is not allowed determine which conservation law(s) is/are violated.

$$\begin{aligned}
\Delta^+ + p &\rightarrow p + p + \gamma \\
\mu^+ + \mu^- &\rightarrow \nu^e + \bar{\nu}_e \\
n + p &\rightarrow \pi^0 + \pi^+ \\
p + \bar{p} &\rightarrow \pi^+ + \pi^- + \pi^0
\end{aligned}$$

Solution

$$\begin{aligned}
\Delta^+ + p &\rightarrow p + p + \gamma \ : \ \text{allowed, em} \\
\mu^+ + \mu^- &\rightarrow \nu^e + \bar{\nu}_e \ : \ \text{allowed, weak} \\
n + p &\rightarrow \pi^0 + \pi^+ \ : \ \text{violates baryon number} \\
p + \bar{p} &\rightarrow \pi^+ + \pi^- + \pi^0 \ : \ \text{allowed, strong}
\end{aligned}$$

Problem 7. A (hypothetical) tetraquark T consists of u and d quarks and/or antiquarks, for example T^{++} is $uu\bar{d}\bar{d}$. Assuming isospin invariance of strong interactions, find the ratio of decay rates of tetra quark with isospin 2

$$\frac{\text{rate of } T^{++} \rightarrow \pi^+\pi^+ \text{ decay}}{\text{rate of } T^{+} \rightarrow \pi^+\pi^0 \text{ decay}}$$

and

$$\frac{\text{rate of } T^0 \to \pi^0\pi^0 \text{ decay}}{\text{rate of } T^0 \to \pi^+\pi^- \text{ decay}}$$

How the second ratio changes if T^0 is in the isospin 1 state?

Solution

Suppose the decay of T particles into π -mesons is described by a certain Hamiltonian $\hat{H}$. We will not need the explicit form of $\hat{H}$ - it is sufficient to know that it is invariant under rotations in the isospin space.

The amplitude of transition of T^{++} state $|I = 2, I_3 = 2\rangle$ to $\pi^+\pi+$ state $|I_3 = 1\rangle|I_3 = 1\rangle$ can be represented as

$$\langle 2,2|\hat{H}|1\rangle|1\rangle \;=\; \langle 2,2|\hat{H}|2,2\rangle\!\rangle\langle\!\langle 2,2\Big||1\rangle|1\rangle$$

where $|\frac{3}{2},\frac{1}{2}\rangle\!\rangle$ denotes the state of $\pi^+\pi+$ pair with total isospin $I = 2$ and projection $I_3 = 2$.

Similarly, the amplitude of the decay of T^+ to $\pi^+\pi^0$ pair can be written down as

$$\langle 2,1|\hat{H}|1\rangle|0\rangle \;=\; \langle 2,1|\hat{H}|2,1\rangle\!\rangle\langle\!\langle 2,1\Big||1\rangle|0\rangle$$

The ratio of decay amplitudes take the form

$$\frac{\langle 2,2|\hat{H}|1\rangle|1\rangle}{\langle 2,2|\hat{H}|1\rangle|0\rangle} \;=\; \frac{\langle 2,2|\hat{H}|2,2\rangle\!\rangle\langle\!\langle 2,2\Big||1\rangle|1\rangle}{\langle 2,1|\hat{H}|2,1\rangle\!\rangle\langle\!\langle 2,1\Big||1\rangle|0\rangle}$$

Since the Hamiltonian $\hat{H}$ is invariant under isospin rotation it matrix elements of $\hat{H}$ can depend only on total isospin I and not on the component I_3. We get

$$\langle 2,2|\hat{H}|2,2\rangle\!\rangle \;=\; \langle 2,1|\hat{H}|2,1\rangle\!\rangle$$

and therefore

$$\frac{\langle 2,2|\hat{H}|1\rangle|1\rangle}{\langle 2,2|\hat{H}|1\rangle|0\rangle} \;=\; \frac{\langle\!\langle 2,2\Big||1\rangle|1\rangle}{\langle\!\langle 2,1\Big||1\rangle|0\rangle}$$

From the table of Clebsch-Gordan coefficients

http://en.wikipedia.org/wiki/Table_of_Clebsch%E2%80%93Gordan_coefficients

we get that

$$|2,1\rangle\!\rangle \;=\; \frac{1}{\sqrt{2}}|1\rangle|0\rangle + \frac{1}{\sqrt{2}}|0\rangle|1\rangle$$

and therefore the ratio of amplitudes is

$$\frac{\langle 2,2|\hat{H}|1\rangle|1\rangle}{\langle 2,2|\hat{H}|1\rangle|0\rangle} \;=\; \frac{\langle 2,2|\hat{H}|2,2\rangle\!\rangle\langle\!\langle 2,2\Big||1\rangle|1\rangle}{\langle 2,1|\hat{H}|2,1\rangle\!\rangle\langle\!\langle 2,1\Big||1\rangle|0\rangle} \;=\; \sqrt{2}$$

The ratio of rates of the decays is proportional to square of the ratio of amplitudes so

$$\frac{\text{rate of } T^{++}\to\pi^+\pi^+ \text{ decay}}{\text{rate of } T^{+}\to\pi^+\pi^0 \text{ decay}} \;=\; \frac{|\langle\!\langle 2,2\Big||1\rangle|1\rangle|^2}{|\langle\!\langle 2,1\Big||1\rangle|0\rangle|^2} \;=\; 2$$

The second ratio is similar

$$\frac{\langle 2,0|\hat{H}|0\rangle|0\rangle}{\langle 2,0|\hat{H}|1\rangle|-1\rangle} \;=\; \frac{\langle 2,0|\hat{H}|2,0\rangle\!\rangle\langle\!\langle 2,0\Big||0\rangle|0\rangle}{\langle 2,0|\hat{H}|2,0\rangle\!\rangle\langle\!\langle 2,0\Big||1\rangle|-1\rangle} \;=\; \frac{\langle\!\langle 2,0\Big||0\rangle|0\rangle}{\langle\!\langle 2,0\Big||1\rangle|-1\rangle}$$

Next, we use the Clebsch-Gordan coefficient

$$|2,0\rangle\!\rangle \;=\; \sqrt{\frac{1}{3}}|1\rangle|-1\rangle - \frac{1}{\sqrt{3}}|0\rangle|0\rangle + \sqrt{\frac{1}{3}}|-1\rangle|1\rangle$$

from the Wiki table and get the result that the ratios are equal.

If T^0 would be in the isospin 1 state

$$\frac{\text{rate of } T^{0}\to\pi^0\pi^0 \text{ decay}}{\text{rate of } T^{0}\to\pi^+\pi^- \text{ decay}} \;=\; \left|\frac{\langle 1,0|\hat{H}|1,0\rangle\!\rangle\langle\!\langle 1,0\Big||0\rangle|0\rangle}{\langle 1,0|\hat{H}|1,0\rangle\!\rangle\langle\!\langle 1,0\Big||0\rangle|0\rangle}\right|^2 \;=\; \frac{|\langle\!\langle 1,0\Big||0\rangle|0\rangle|^2}{|\langle\!\langle 1,0\Big||1\rangle|-1\rangle|^2} \;=\; 0$$

because

$$|1,0\rangle\!\rangle \;=\; \sqrt{\frac{1}{2}}|1\rangle|-1\rangle - \frac{1}{\sqrt{2}}|-1\rangle|1\rangle$$

so $\langle 1,0\Big||0\rangle|0\rangle$ vanishes.

Problem 8. Draw possible Feynman diagram(s) for

$$K^+ + \pi^- \;\to\; \pi^+ + \pi^-$$

$$\Omega^- + \gamma \;\to\; \Xi^0 + K^-$$

$$\bar{n} + p \;\to\; \pi^0 + \pi^+$$

All problems have equal weight.