Problem 10.2 solution.

Let us start with Problem 10.2a.

Because of the Gell-Mann relation $Q = I_3 + \frac{S}{2}$ the third component of isospin for the K^{*++} particle is $I_3 = \frac{3}{2}$ and for K^{*+} particle $I_3 = \frac{1}{2}$. (The notation K is reserved for mesons with $S = 1$).

Suppose the decay of K^* particles into $K-\pi$ pairs is described by a certain Hamiltonian $\hat{H}$. We will not need the explicit form of $\hat{H}$ - it is sufficient to know that it is invariant under rotations in the isospin space.

The amplitude of transition of K^{*++} state $|I = \frac{3}{2}, I_3 = \frac{3}{2}\rangle$ to $K^+\pi+$ state $|I_3 = \frac{1}{2}\rangle|I_3 = 1\rangle$ can be represented as

$$\langle\frac{3}{2},\frac{3}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle = \langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{3}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{3}{2}\Big||\frac{1}{2}\rangle|1\rangle$$

where $|\frac{3}{2},\frac{1}{2}\rangle\rangle$ denotes the state of $K^+\pi+$ pair with total isospin $I = \frac{3}{2}$ and projection $I_3 = \frac{3}{2}$.

Similarly, the amplitude of the decay of K^* to $K^+\pi^0$ pair can be written down as

$$\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{1}{2}\rangle|0\rangle = \langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{1}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle$$

The ratio of decay amplitudes take the form

$$\frac{\langle\frac{3}{2},\frac{3}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle}{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{1}{2}\rangle|0\rangle} = \frac{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{3}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{3}{2}\Big||\frac{1}{2}\rangle|1\rangle}{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{1}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle}$$

Since the Hamiltonian $\hat{H}$ is invariant under isospin rotation it matrix elements of $\hat{H}$ can depend only on total isospin I and not on the component I_3. We get

$$\langle\frac{3}{2},\frac{3}{2}|\hat{H}|\frac{3}{2},\frac{3}{2}\rangle\rangle = \langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{1}{2}\rangle\rangle$$

and therefore

$$\frac{\langle\frac{3}{2},\frac{3}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle}{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle} = \frac{\langle\langle\frac{3}{2},\frac{3}{2}\Big||\frac{1}{2}\rangle|1\rangle}{\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle}$$

From the table of Clebsch-Gordan coefficients
http://en.wikipedia.org/wiki/Table_of_Clebsch%E2%80%93Gordan_coefficients

we get that

$$|\frac{3}{2},\frac{1}{2}\rangle\rangle \;=\; \sqrt{\frac{2}{3}}|\frac{1}{2}\rangle|0\rangle + \frac{1}{\sqrt{3}}|-\frac{1}{2}\rangle|1\rangle$$

and therefore the ratio of amplitudes is

$$\frac{\langle\frac{3}{2},\frac{3}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle}{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{1}{2}\rangle|1\rangle} \;=\; \frac{\langle\langle\frac{3}{2},\frac{3}{2}\Big||\frac{1}{2}\rangle|1\rangle}{\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle} \;=\; \frac{1}{\sqrt{\frac{2}{3}}} \;=\; \frac{\sqrt{3}}{\sqrt{2}}$$

The ratio of rates of the decays is proportional to square of the ratio of amplitudes so

$$\frac{\text{rate of } K^{*++}\to K^{+}\pi^{+} \text{ decay}}{\text{rate of } K^{*+}\to K^{+}\pi^{0} \text{ decay}} \;=\; \frac{|\langle\langle\frac{3}{2},\frac{3}{2}\Big||\frac{1}{2}\rangle|1\rangle|^2}{|\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle|^2} \;=\; \frac{3}{2}$$

The solution of the rest of 10.2 problem is similar. We use the Clebsch-Gordan coefficient

$$|\frac{3}{2},-\frac{1}{2}\rangle\rangle \;=\; \sqrt{\frac{2}{3}}|-\frac{1}{2}\rangle|0\rangle + \frac{1}{\sqrt{3}}|\frac{1}{2}\rangle|-1\rangle$$

from the Wiki table and get

$$\frac{\text{rate of } K^{*+}\to K^{+}\pi^{0} \text{ decay}}{\text{rate of } K^{*+}\to K^{0}\pi^{+} \text{ decay}} \;=\; \left|\frac{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{1}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle}{\langle\frac{3}{2},\frac{1}{2}|\hat{H}|\frac{3}{2},\frac{1}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle}\right|^2 \;=\; \frac{|\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle|^2}{|\langle\langle\frac{3}{2},\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle|^2} \;=\; 2$$

$$\frac{\text{rate of } K^{*-}\to K^{0}\pi^{-} \text{ decay}}{\text{rate of } K^{*0}\to K^{+}\pi^{-} \text{ decay}} \;=\; \left|\frac{\langle\frac{3}{2},-\frac{3}{2}|\hat{H}|\frac{3}{2},-\frac{3}{2}\rangle\rangle\langle\langle\frac{3}{2},-\frac{3}{2}\Big||-\frac{1}{2}\rangle|-1\rangle}{\langle\frac{3}{2},-\frac{1}{2}|\hat{H}|\frac{3}{2},-\frac{1}{2}\rangle\rangle\langle\langle\frac{3}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|-1\rangle}\right|^2 \;=\; \frac{|\langle\langle\frac{3}{2},-\frac{3}{2}\Big||-\frac{1}{2}\rangle|-1\rangle|^2}{|\langle\langle\frac{3}{2},-\frac{1}{2}\Big||\frac{1}{2}\rangle|-1\rangle|^2} \;=\; 3$$

Problem 10.2b.
The solution is similar but now we need the Clebsch-Gordan coefficients

$$|\frac{1}{2},\frac{1}{2}\rangle\rangle \;=\; \sqrt{\frac{2}{3}}|-\frac{1}{2}\rangle|1\rangle + \frac{1}{\sqrt{3}}|\frac{1}{2}\rangle|0\rangle$$

$$|\frac{1}{2},-\frac{1}{2}\rangle\rangle \;=\; -\sqrt{\frac{2}{3}}|\frac{1}{2}\rangle|-1\rangle + \frac{1}{\sqrt{3}}|-\frac{1}{2}\rangle|0\rangle$$

The state sK^{*++} and K^{*-} with isospin $\frac{1}{2}$ contradict Gell-Mann-Nishijima relation so we are left with decays of K^{*+} with $I_3 = \frac{1}{2}$ and K^{*0} with $I_3 = -\frac{1}{2}$.
The ratios are:

$$\frac{\text{rate of } K^{*+} \to K^+\pi^0 \text{ decay}}{\text{rate of } K^{*+} \to K^0\pi^+ \text{ decay}} = \left|\frac{\langle\frac{1}{2},\frac{1}{2}|\hat{H}|\frac{1}{2},\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle}{\langle\frac{1}{2},\frac{1}{2}|\hat{H}|\frac{1}{2},\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle}\right|^2 = \frac{|\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle|^2}{|\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle|^2} = \frac{1}{2}$$

$$\frac{\text{rate of } K^{*+} \to K^+\pi^0 \text{ decay}}{\text{rate of } K^{*0} \to K^+\pi^- \text{ decay}} = \left|\frac{\langle\frac{1}{2},\frac{1}{2}|\hat{H}|\frac{1}{2},\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle}{\langle\frac{1}{2},-\frac{1}{2}|\hat{H}|\frac{1}{2},-\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle}\right|^2 = \frac{|\langle\!\langle\frac{1}{2},\frac{1}{2}\Big||\frac{1}{2}\rangle|0\rangle|^2}{|\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||\frac{1}{2}\rangle|-1\rangle|^2} = \frac{1}{2}$$

$$\frac{\text{rate of } K^{*0} \to K^0\pi^0 \text{ decay}}{\text{rate of } K^{*0} \to K^+\pi^- \text{ decay}} = \left|\frac{\langle\frac{1}{2},-\frac{1}{2}|\hat{H}|\frac{1}{2},-\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||-\frac{1}{2}\rangle|0\rangle}{\langle\frac{1}{2},-\frac{1}{2}|\hat{H}|\frac{1}{2},-\frac{1}{2}\rangle\!\rangle\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||-\frac{1}{2}\rangle|1\rangle}\right|^2 = \frac{|\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||-\frac{1}{2}\rangle|0\rangle|^2}{|\langle\!\langle\frac{1}{2},-\frac{1}{2}\Big||\frac{1}{2}\rangle|-1\rangle|^2} = \frac{1}{2}$$