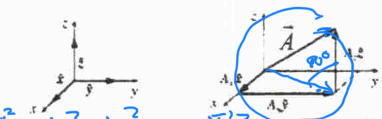
Welcome to

Introduction
to

Electromagnetic Theory

Coordinate Systems

Cartesian Coordinates



$A_{xy}^2 = A_x^2 + A_y^2$ $|\vec{A}|^2 = A_{xy}^2 + A_z^2$

A vector is specified by its ***components*** along 3 perpendicular axes.

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ — "unit vector"
- magnitude 1
- points along axis

Note that $\vec{A}$ is the sum of the 3 vectors which point in the x,y,z direction.

The magnitude of $\vec{A} = |\vec{A}|$ which is

$$|\vec{A}| = \left[A_x^2 + A_y^2 + A_z^2\right]^{1/2}$$

Vector operations in Cartesian coordinates

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad \hat{i} \to \hat{x}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \qquad \hat{j} \to \hat{y}$$
$$\hat{k} \to \hat{z}$$

Addition: Simply add components

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Subtraction: Add the inverse vector

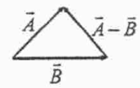
$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$
$$\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

Note: - addition is commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- addition is associative: $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

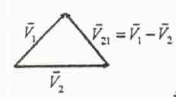
The sum or difference of 2 vectors is itself a vector.

And the magnitude of the difference of 2 vectors is:

$$|(\vec{A} - \vec{B})| = \left[(A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2\right]^{\frac{1}{2}}$$



Notation: Sometimes we will write



Multiplication:

Scalar times vector: $\Rightarrow a\vec{A}$

$a>0$ direction unchanged
$a<0$ direction reversed

Scalar multiplication is distributive:

$$a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$$

Vector times vector:

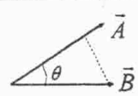
- dot product (scalar product)

$$\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}|\cos\theta$$

The dot product is a scalar and

is commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

and distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$



θ is the angle between $\vec{A}$ and $\vec{B}$

We can think of $B\cos\theta$ as the "projection" of $\vec{B}$ onto the direction of $\vec{A}$

So the dot product is a measure of how much one vector points along the direction of the other vector.

For $\vec{A}, \vec{B}$ parallel $\quad \vec{A} \cdot \vec{B} = AB \quad (\cos 0 = 1)$

$\vec{A}, \vec{B}$ perpendicular $\quad \vec{A} \cdot \vec{B} = 0 \quad \left(\cos\frac{\pi}{2} = 0\right)$

Since $\vec{A} \cdot \vec{A} = A^2$ we see that the magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$

What about unit vectors:

$$\hat{x}\cdot\hat{x}=\hat{y}\cdot\hat{y}=\hat{z}\cdot\hat{z}=1$$

$$\hat{x}\cdot\hat{y}=\hat{y}\cdot\hat{z}=\hat{x}\cdot\hat{z}=0$$

From which it follows

$$\vec{A}\cdot\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\cdot\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)$$
$$=A_xB_x+A_yB_y+A_zB_z$$

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$\times$ cross product (vector product)

The cross product of 2 vectors is a vector whose direction is $\perp$ to the plane containing the 2 initial vectors.

$$\boxed{\vec{D}=\vec{A}\times\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta\hat{n}}$$

$\hat{n}$ is unit vector $\perp$ to plane containing $\vec{A}$ and $\vec{B}$

There are two possible directions for the resulting vector
– use "right hand rule":
Curl fingers (of your right hand!) from $\vec{A}$ to $\vec{B}$,
then $\hat{n}$ is the direction of your thumb.

The cross product is distributive:

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

But, it is not commutative:

$$\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$$

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Unit vectors: $\hat{x}\times\hat{x}=\hat{y}\times\hat{y}=\hat{z}\times\hat{z}=0$

$$\hat{x}\times\hat{y}=\hat{z}\qquad\hat{y}\times\hat{x}=-\hat{z}$$
$$\hat{y}\times\hat{z}=\hat{x}\qquad\hat{z}\times\hat{y}=-\hat{x}$$
$$\hat{z}\times\hat{x}=\hat{y}\qquad\hat{x}\times\hat{z}=-\hat{y}$$

So: $\vec{A}\times\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\times\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)$

$$=A_xB_x\hat{x}\times\hat{x}+A_xB_y\hat{x}\times\hat{y}+A_xB_z\hat{x}\times\hat{z}+A_yB_x\hat{y}\times\hat{x}+A_yB_y\hat{y}\times\hat{y}+A_yB_z\hat{y}\times\hat{z}+A_zB_x\hat{z}\times\hat{x}+A_zB_y\hat{z}\times\hat{y}+A_zB_z\hat{z}\times\hat{z}$$
$$=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}$$

Note that the cross product can be written as a determinant:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

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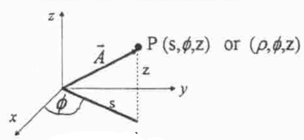
Cylindrical Coordinates



Note: $\hat{s}$ and $\hat{\phi}$ depend of ϕ.

Unit vectors:

$$\hat{s}=\cos\phi\hat{x}+\sin\phi\hat{y}$$
$$\hat{\phi}=-\sin\phi\hat{x}+\cos\phi\hat{y}$$
$$\hat{z}=\hat{z}$$

$$\vec{A}=A_z\hat{z}+A_s\hat{s}$$

The coordinate transformation to Cartesian coordinates:

$$x=s\cos\phi$$
$$y=s\sin\phi$$
$$z=z$$

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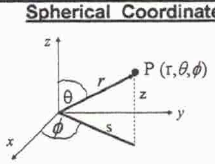
Spherical Coordinates



θ - polar angle
ϕ - azimuthal angle

$$\hat{\theta}\cdot\hat{\theta}=\hat{r}\cdot\hat{r}=\hat{\varphi}\cdot\hat{\varphi}=1$$

$$\vec{A}=A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi}$$

$$\hat{r}\cdot\hat{\theta}=0$$
$$\hat{\theta}\cdot\hat{\varphi}=0$$
$$\hat{r}\cdot\hat{\varphi}=0$$

Unit vectors:

$$\hat{r}=\sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}+\cos\theta\hat{z}$$
$$\hat{\theta}=\cos\theta\cos\phi\hat{x}+\cos\theta\sin\phi\hat{y}-\sin\theta\hat{z}$$
$$\hat{\phi}=-\sin\phi\hat{x}+\cos\phi\hat{y}$$

The coordinate transformation to Cartesian coordinates:

$$x=r\sin\theta\cos\phi$$
$$y=r\sin\theta\sin\phi$$
$$z=r\cos\theta$$

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Uses of vectors

One of the uses of a vector is a "locator"

Consider a point P(x,y,z). The position vector is defined:

$$\vec{r}\equiv x\hat{x}+y\hat{y}+z\hat{z}$$

It is a vector which points from the origin to point P.

Clearly, it has magnitude

$$r=\left[x^2+y^2+z^2\right]^{1/2}$$

And the unit vector is simply the normalized $\vec{r}$ vector

$$\hat{r}=\frac{\vec{r}}{r}=\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\left[x^2+y^2+z^2\right]^{1/2}}$$

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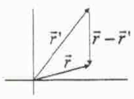
What about the difference or "separation" of two vectors?



$$\vec{\mathfrak{R}} = \vec{r} - \vec{r}'$$

We will encounter $\vec{\mathfrak{R}}$ frequently since one point is often a "source" and the other a "point of interest" (to determine the effect of the source).

A vector can also be considered a function to represent a physical quantity associated with a point (like a field).

We would write $\vec{A}(\vec{r})$ $\vec{A}$ - vector function
$\vec{r}$ - point location

Of course, our function could also be a scalar: $f(\vec{r})$

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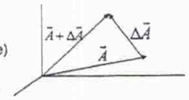
Calculus

Derivatives

Recall that a derivative measures how fast a function changes with respect to a change in a independent variable.

For example, a rate (change w.r.t. time)



$$\Delta\vec{A} = \Delta A_x\hat{x} + \Delta A_y\hat{y} + \Delta A_z\hat{z}$$

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\Delta A_x\hat{x} + \Delta A_y\hat{y} + \Delta A_z\hat{z}}{\Delta t}$$

$$= \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z}$$

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What about spatial derivatives?

Scalar functions

- Consider a one-dimensional function $f(x)$.
  The derivative $\frac{df}{dx}$ is just the slope of f vs. x.

What about higher dimensions?

- Consider $f(x,y)$ which is continuous and differentiable.
  How does f changes if you move $d\vec{l}$?

$$d\vec{l} = dx\hat{x} + dy\hat{y} \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

But, it can be written:

$$\boxed{df = \vec{A}\cdot d\vec{l}} \qquad \vec{A} = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

The vector $\vec{A}$ is known as the "**gradient**"

$$\boxed{\vec{A} = grad\ f \equiv \vec{\nabla} f} \qquad \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

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- For 3 dimensions:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \quad \Rightarrow \quad \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

"del" operator   \nabla

What is the magnitude of $\vec{\nabla} f$?

$$\left|\vec{\nabla} f\right| = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]^{1/2}$$

$$df = \vec{A}\cdot d\vec{l} = \vec{\nabla} f\cdot d\vec{l} = \left|\vec{\nabla} f\right|\left|d\vec{l}\right|\cos\theta$$

Clearly df is maximum when $\cos\theta = 1$ $(\theta = 0)$
i.e. $d\vec{l}$ is in the direction of $\vec{\nabla} f$.

The gradient is a vector whose magnitude and direction are those of the max (spatial) change in f.

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Problem 1.12. The height of a certain hill (in feet) is given by

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is the distance (in miles) north, x the distance east of a city.

(a) Where is the top of the hill located?
(b) How high is the hill?
(c) How steep is the slope (feet/mile) at a point 1 mile north and 1 mile east of the city. In what direction is the slope steepest, at that point?

$$\vec{\nabla} h = \frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y} = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

(a) At the top, $\vec{\nabla} h = 0$ $\therefore$

$2y - 6x - 18 = 0 \quad \Rightarrow \quad -6x + 2y - 18 = 0$

$2x - 8y + 28 = 0 /\times 3 \quad \Rightarrow \quad 6x - 24y + 84 = 0$

$-22y + 66 = 0$

$y = 3 \quad x = -2$

The top is 3 miles north and 2 miles west of the city.

(b) $h(-2,3) = 10(-12 - 12 - 36 + 36 + 84 + 12)ft = 720\,ft$

(c) $\vec{\nabla} h(1,1) = 10(2 - 6 - 18)\hat{x} + 10(2 - 8 + 28)\hat{y} = -220\hat{x} + 220\hat{y}$

$\left|\vec{\nabla} h(1,1)\right| = \sqrt{220^2 + 220^2} = 220\sqrt{2}\,ft/mile \cong 311\,ft/mile$

Direction: NORTHEAST

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Vector functions

$\vec{\nabla}$ is a vector operator. When it operates on another vector, there are 2 possible operations:
- dot product
- cross product

Consider **Dot product**:

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

a scalar

$\vec{\nabla}\cdot$ is known as "**divergence**"

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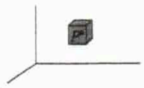
What does the divergence measure?

Think of an infinitesimal volume element surrounding a point *P*.



At every point in space, there is a value for the vector function (vector field).

For example, if we calculate the <u>flux</u> through the surface of the volume element:

$$\Phi = \int_s \vec{A}\cdot d\vec{a}$$

We can show

$$\Phi = \int_\tau \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau$$

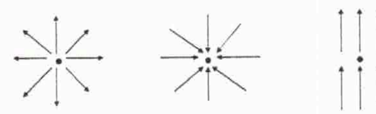
volume

Divergence is the measure how the vector function spreads out. 19

Therefore, $\vec{\nabla}\cdot\vec{A}$ is clearly the **<u>outgoing flux per unit volume</u>**.

$$\int_s \vec{A}\cdot d\vec{a} = \int_\tau \vec{\nabla}\cdot\vec{A}\, d\tau$$

This is known as: Gauss' Theorem
Green's Theorem
divergence Theorem



positive divergence "source" | negative divergence "sink" | zero divergence

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<u>Problem 1.16.</u> Sketch the vector function $\vec{v} = \frac{\hat{r}}{r^2}$, and compute its divergence. Explain the answer.

$$\vec{v} = \frac{\hat{r}}{r^2} = \frac{\frac{\vec{r}}{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{x\hat{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{y\hat{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{z\hat{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) =$$

$$= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - x\frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$+ \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - y\frac{3}{2}\frac{2y}{(x^2+y^2+z^2)^{\frac{5}{2}}} +$$

$$+ \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - z\frac{3}{2}\frac{2z}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\vec{\nabla}\cdot\vec{v} = 3r^{-3} - 3r^{-5}\left(x^2+y^2+z^2\right) = 3r^{-3} - 3r^{-3} = 0.$$

As $r\to 0$, $v\to\infty$, where divergence is also infinite. So divergence is 0 everywhere, except at origin, where it is infinite. 21

<u>Cross Product (Curl)</u> $\vec{\nabla}\times\vec{A} = curl$

Consider the cross product
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

What does this correspond to? Let's introduce integral calculus.

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<u>Problem 1.18.</u> Calculate the curl of the vector function in problem 1.15c

$$\vec{v} = y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy+z^2) & 2yz \end{vmatrix} =$$

$$= \hat{x}\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy+z^2)\right) - \hat{y}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(y^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}(y^2)\right)$$

$$= \hat{x}(2z-2z) - \hat{y}(0-0) + \hat{z}(2y-2y) = 0$$

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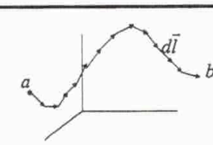
<u>Integral Calculus</u>



Consider a <u>line integral</u>

$$\int_a^b \vec{A}\cdot d\vec{l}$$

If your ending point and beginning point are the same, the path is said to be <u>closed</u> and the integral is written: $\oint \vec{A}\cdot d\vec{l}$

If $\oint \vec{A}\cdot d\vec{l} = 0$ for <u>any</u> closed path, the vector $\vec{A}$ is said to be **<u>conservative</u>**.

In general, though $\oint \vec{A}\cdot d\vec{l} \neq 0$

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Consider 2-D example: $\vec{A}\cdot d\vec{l} = A_x dx + A_y dy$

$$\Rightarrow \quad \oint \vec{A}\cdot d\vec{l} = \oint A_x dx + \oint A_y dy$$

We can show (for infinitesimal path) $\oint A_x dx = -\frac{\partial A_x}{\partial y} dydx$

$\oint A_y dy = \frac{\partial A_y}{\partial x} dxdy$

$$\oint \vec{A}\cdot d\vec{l} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dxdy$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) da$$

We can then extend this to 3D $\oint \vec{A}\cdot d\vec{l} = \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}$

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For arbitrary path we can extend this to

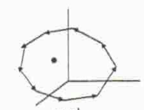
$$\oint \vec{A}\cdot d\vec{l} = \int_s \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}$$

surface

path that bounds surface

This is known as "Stokes Theorem"

We can see that $\boxed{\vec{\nabla}\times\vec{A}}$ is then a measure of the twist or **"curl"** of a vector field about a point.



curl

no curl

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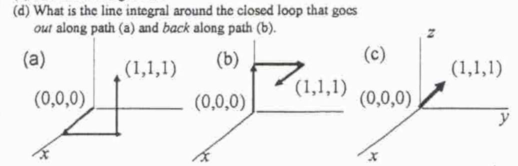
Problem 1.28. Calculate the line integral of the function $\vec{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$ from the origin to the point (1,1,1) by three different routes:

(a) (0,0,0)→(1,0,0)→(1,1,0)→(1,1,1)
(b) (0,0,0)→(0,0,1)→(0,1,1)→(1,1,1)
(c) The direct straight line.
(d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b).



(a) $(0,0,0)\to(1,0,0) \equiv x:\ 0\to 1, y = z = 0 \Rightarrow d\vec{l} = dx\hat{x}; \int \vec{v}\cdot d\vec{l} = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$

$(1,0,0)\to(1,1,0) \equiv x = 1, y:\ 0\to 1, z = 0 \Rightarrow d\vec{l} = dy\hat{y}; \vec{v}\cdot d\vec{l} = 2yzdy = 0; \int \vec{v}\cdot d\vec{l} = 0$

$(1,1,0)\to(1,1,1) \equiv x = y = 1, z:\ 0\to 1 \Rightarrow d\vec{l} = dz\hat{z}; \int \vec{v}\cdot d\vec{l} = \int_0^1 y^2 dz = (z)_0^1 = 1$

$\int \vec{v}\cdot d\vec{l} = \frac{1}{3} + 0 + 1 = \frac{4}{3}$

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Problem 1.28 (Continued)

(b) $(0,0,0)\to(0,0,1) \equiv x = y = 0, z: 0\to 1; d\vec{l} = dz\hat{z}; \vec{v}\cdot d\vec{l} = y^2dz = 0; \int \vec{v}\cdot d\vec{l} = 0$

$(0,0,1)\to(0,1,1) \equiv x = 0, y: 0\to 1, z = 1; \int \vec{v}\cdot d\vec{l} = \int_0^1 2ydy = \left(y^2\right)_0^1 = 1$

$(0,1,1)\to(1,1,1) \equiv x: 0\to 1, y = z = 1; \int \vec{v}\cdot d\vec{l} = \int_0^1 x^2dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$

$\int \vec{v}\cdot d\vec{l} = 0 + 1 + \frac{1}{3} = \frac{4}{3}$

(c) $x = y = z: 0\to 1; dx = dy = dz$

$\vec{v}\cdot d\vec{l} = x^2dx + 2yzdy + y^2dz = x^2dx + 2x^2dx + x^2dx = 4x^2dx$

$\int \vec{v}\cdot d\vec{l} = \int_0^1 4x^2dx = \left(\frac{4x^3}{3}\right)_0^1 = \frac{4}{3}$

(d) $\oint \vec{v}\cdot d\vec{l} = \int_{(a)} \vec{v}\cdot d\vec{l} - \int_{(b)} \vec{v}\cdot d\vec{l} = \frac{4}{3} - \frac{4}{3} = 0$

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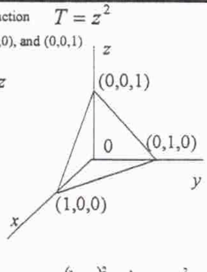
Problem 1.30. Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at (0,0,0), (1,0,0), (0,1,0), and (0,0,1)

$$\int_V T d\tau = \int_V z^2 dxdydz = \int z^2 \left[\int \left(\int dx\right) dy\right] dz$$



Tetrahedron: $x + y + z = 1$

$\Rightarrow x: 0\to(1-y-z), y: 0\to(1-z), z: 0\to 1$

$$\int dx = \int_0^{1-y-z} dx = 1 - y - z$$

$$\int\left(\int dx\right)dy = \int_0^{1-z}(1-y-z)dy = \int_0^{1-z}(1-z)dy - \int_0^{1-z} ydy = (1-z)^2 - \frac{(1-z)^2}{2} = \frac{1}{2} - z + \frac{z^2}{2}$$

$$\int z^2\left[\int\left(\int dx\right)dy\right]dz = \int_0^1 z^2\left(\frac{1}{2} - z + \frac{z^2}{2}\right)dz = \int_0^1\left(\frac{z^2}{2} - z^3 + \frac{z^4}{2}\right)dz = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}$$

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The gradient $(\vec{\nabla}f)$, divergence $(\vec{\nabla}\cdot\vec{A})$, and curl $(\vec{\nabla}\times\vec{A})$ are the first derivatives.

Second Derivatives: $\vec{\nabla}$ acting on a gradient.

Divergence of a gradient

$$\vec{\nabla}\cdot\left(\vec{\nabla}f\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is called the **"Laplacian"** and is usually written $\boxed{\nabla^2 f}$

Laplacian of the scalar function

Curl of the gradient $\boxed{\vec{\nabla}\times\left(\vec{\nabla}f\right) = 0}$

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Gradient of the divergence $\boxed{\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)}$ (not the Laplacian)

Divergence of curl $\boxed{\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0}$

Curl of the curl $\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}}$

$$\underbrace{\nabla^2\vec{A}}_{\text{Laplacian of the vector function}}=\left(\nabla^2 A_x\right)\hat{x}+\left(\nabla^2 A_y\right)\hat{y}+\left(\nabla^2 A_z\right)\hat{z}$$

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Cylindrical and Spherical coordinates

We have been working in Cartesian coordinates where the expressions are fairly simple. In other systems, the expressions are more cumbersome.

For example, the curl operator in spherical coordinates is:

$$\vec{\nabla}\times\vec{A}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\Phi)-\frac{\partial A_\theta}{\partial\Phi}\right]\hat{r}$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\Phi}-\frac{\partial}{\partial r}(rA_\Phi)\right]\hat{\theta}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\Phi}$$

Use coordinates which exploit the symmetry of the problem!

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Dirac Delta Function

Divergence of $\frac{\hat{r}}{r^2}$

Consider $\vec{v}=\frac{\hat{r}}{r^2}$

By definition, should have large positive divergence.

Yet, divergence in spherical coordinates:

$$\vec{\nabla}\cdot\vec{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2|\vec{v}|\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)=0$$

Paradox!

Outgoing flux per unit volume $\neq 0$ (see Figure)

Recall divergence theorem: $\oint\vec{v}\cdot d\vec{a}=\int_{volume}\vec{\nabla}\cdot\vec{v}\,d\tau$

Evaluate separately right and left side:

Right side: $\int_{volume}\vec{\nabla}\cdot\vec{v}\,d\tau=\int_{volume}0\cdot d\tau=0$

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Consider sphere with radius *R*.

Left side: $\oint\vec{v}\cdot d\vec{a}=\int\left(\overbrace{\frac{1}{R^2}\hat{r}}^{\vec{v}}\right)\cdot\overbrace{R^2\sin\theta d\theta d\phi\hat{r}}^{d\vec{a}}$ in spherical coordinates.

$\hat{r}\cdot\hat{r}=1$

$$=\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\phi=2\cdot2\pi=4\pi$$

Left side $\neq$ Right side ?!

$\boxed{r^2\cdot\frac{1}{r^2}\neq1 \quad when \quad r=0}$

To resolve this problem, one can introduce

"**Dirac Delta Function**" $f(r)=0$ *in whole space* (Except r=0)

$$\int_{-\infty}^{\infty}f(r)dr = finite \quad number$$

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One-Dimensional Dirac Delta Function

Definitions: $\boxed{\delta(x)=\begin{cases}0 & x\neq0\\ \infty & x=0\end{cases}}$ and $\boxed{\int_{-\infty}^{\infty}\delta(x)dx=1}$

Properties:

$$f(x)\delta(x)=f(0)\delta(x)$$

$$\int f(x)\delta(x)dx=\int f(0)\delta(x)dx$$

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx=f(0)\int_{-\infty}^{\infty}\delta(x)dx=f(0)$$

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

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Three-Dimensional Dirac Delta Function

$$\delta^3(\vec{r})=\delta(x)\delta(y)\delta(z)$$

$$\int_{all\ space}\delta^3(\vec{r})d\tau=1$$

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)=4\pi\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\left(\frac{\hat{\Re}}{\Re^2}\right)=4\pi\delta^3(\vec{\Re})$$

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