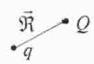
(Chapter 2) Electrostatics

One of the fundamental properties of matter is "charge"
- Charge is quantized
- Charge comes in the two types (- and +, by convention)
- Net charge cannot be created or destroyed (conservation of charge)

To begin, lets consider the case where the charges are not moving
⇒ **electrostatic**

Consider two charges: q and Q

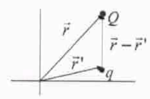


The force on Q due to q is:

Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\Re^2}\hat{\Re}$$

Permitivity of free space: $\varepsilon_0 = 8.85\times10^{-12}\,C^2/Nm^2$



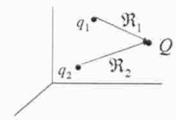
$$\vec{\Re} = \vec{r} - \vec{r}\,'$$

$$\hat{\Re} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

What about multiple charges? ⇒ Use the principle of superposition:

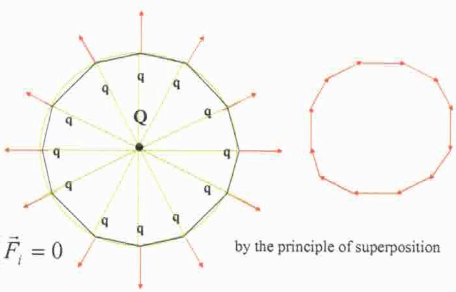
The interaction between any two charges is not affected by the presence of a third charge.

$$\Rightarrow \quad \vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 + \dots$$



Problem 2.1. Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face).
(a) What is the net force on a test charge Q at the center?
(b) Suppose one of the 12 q's is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.

(a)

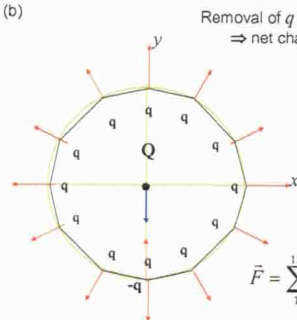


$$\vec{F} = \sum_{1}^{12}\vec{F}_i = 0$$

by the principle of superposition

(b)

Removal of q is equivalent to addition of $-q$
⇒ net charge at "6 o'clock" is 0.



$$\vec{F}_6 = \frac{1}{4\pi\varepsilon_0}\frac{-qQ}{r^2}\hat{j}$$

$$\vec{F} = \sum_{1}^{12}\vec{F}_q + \vec{F}_6 = 0 - \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}$$

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{j}$$

Electric Field

Recall Coulomb Law: The two charges q and Q are not touching. There is no contact force. So how is there a force?

One way to answer this question is that charge q **disturbs** the space around it.

This disturbance is known as field, $\vec{E}$.

Charge Q then interacts with the field created by q and experiences a force given by:

$$\vec{F} = Q\vec{E}$$

Using Coulomb's Law:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re^2}\hat{\Re}$$

For multiple charges, simply add

1. Charges discrete → sum
2. Charges continuous → integrate

1. $$\vec{F}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}Q\sum_{i=1}^{n}\frac{q_i}{\Re_i^2}\hat{\Re}_i$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\Re_i^2}\hat{\Re}_i$$

2. $$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\Re^2}\hat{\Re}$$

dq depends on the geometry of the system.

Continuous Charge Distribution

(a) Line Charges $dq \to \lambda dl'$ (λ: charge/unit length)

(b) Surface charges $dq \to \sigma da'$ (σ: charge/unit surface area)

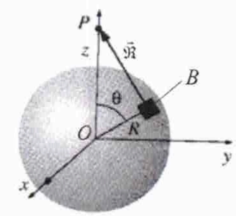
(c) Volume charges $dq \to \rho d\tau'$ (ρ: charge/unit volume)

Note: Be careful about $\hat{\mathfrak{R}}$!

$\hat{\mathfrak{R}}$ has magnitude 1, but points in a direction that depends on the location of dq and field point defined by $\vec{r}$.

Problem 2.7. Find $\vec{E}(z)$ a distance z from the center of a spherical surface of radius R, which carries a uniform surface charge density σ.

Case: (1) $z<R$
(2) $z>R$

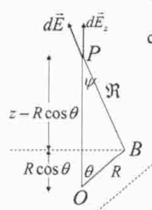


By symmetry $\vec{E}$ is along $\hat{z}$.
Let us work in spherical coordinates.

$dq = \sigma da = \sigma R^2 \sin\theta d\theta d\phi$ (spherical coordinates)

$\vec{\mathfrak{R}} = \vec{R} - \vec{z}$

From the Law of Cosinus: $\mathfrak{R}^2 = R^2 + z^2 - 2Rz\cos\theta$



$$\cos\psi = \frac{z - R\cos\theta}{\mathfrak{R}}$$

By symmetry:

$$E_{x(total)} = E_{y(total)} = 0$$

$$E_z = \int_{surface} dE_z = \int_{surface} dE\cos\psi$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\int \frac{d\theta d\phi\left[\sigma R^2 \sin\theta\right]}{\left(R^2 + z^2 - 2Rz\cos\theta\right)}\overbrace{\left[\frac{z - R\cos\theta}{\mathfrak{R}}\right]}^{\cos\psi}$$

$$\int d\phi = 2\pi$$

$$E_z = \frac{2\pi R^2\sigma}{4\pi\varepsilon_0}\int_0^{\pi} d\theta \frac{(z - R\cos\theta)\sin\theta}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}}$$

Let $u = \cos\theta \Rightarrow du = -\sin\theta d\theta$ $\quad \theta = 0 \quad u = +1$; $\quad \theta = \pi \quad u = -1$

$$E_z = \frac{2\pi R^2\sigma}{4\pi\varepsilon_0}\int_{-1}^{1} du \frac{z - Ru}{\left(R^2 + z^2 - 2Rzu\right)^{3/2}}$$

Look up integral in the Tables!

$$= \frac{2\pi R^2\sigma}{4\pi\varepsilon_0}\left[\frac{1}{z^2}\frac{zu - R}{\left[R^2 + z^2 - 2Rzu\right]^{1/2}}\right]_{-1}^{1}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2\sigma}{z^2}\left[\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right]$$

$$\left\{\begin{matrix} z>R \Rightarrow \left[\frac{z-R}{z-R} + \frac{z+R}{z+R} = 2\right] \\ z<R \Rightarrow \left[-\frac{z-R}{z-R} + \frac{z+R}{z+R} = -1+1 = 0\right]\end{matrix}\right\}$$

$z > R$ $\quad E_z = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2\sigma}{z^2}[2] = \frac{1}{4\pi\varepsilon_0}\frac{\overbrace{4\pi R^2\sigma}^{q}}{z^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}$

As if the q is in the center of the sphere.

$z<R$ $\quad E_z = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R^2\sigma}{z^2}[0] = 0$

Problem 2.8. Find $\vec{E}(\vec{r})$ inside and outside a sphere of radius R which carries a uniform volume charge density ρ.

$$q = \rho V = \rho \tfrac{4}{3}\pi R^3$$

- outside $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{\rho R^3}{3r^2\varepsilon_0}\hat{r}$ $\quad$ As if the q is in the center of the sphere.

- inside $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{inside}}{r^2}\hat{r}$

$$\frac{Q_{inside}}{V_{inside}} = \frac{q}{V}$$

$$Q_{inside} = \frac{(4/3)\pi r^3}{(4/3)\pi R^3}q = \frac{r^3}{R^3}q = \frac{4}{3}\pi r^3\rho$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r} = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

Divergence of Electric Field

We can always calculate $\vec{E}$ by direct integration. But … for problems with symmetry there is an easier way.

Consider the electric field flux through a surface: $\Phi_E = \int_s \vec{E}\cdot d\vec{a}$

For a point charge: $\oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\cdot\left(r^2\sin\theta d\theta d\phi\hat{r}\right) = \frac{q}{\varepsilon_0}$

This is the general result:

The flux through any surface enclosing the charge is $\frac{q}{\varepsilon_0}$.

For multiple charges (by superposition)

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$$

Gauss' Law

If we use the divergence theorem

$$\int_S \vec{E}\cdot d\vec{a} = \int_V \left(\vec{\nabla}\cdot\vec{E}\right)d\tau \qquad \left(= \frac{Q_{enclosed}}{\varepsilon_0}\right)$$

and we write $Q_{encl} = \int_V \rho\, d\tau$

$$\Rightarrow \int_V \left(\vec{\nabla}\cdot\vec{E}\right)d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

Using Gauss' Law greatly simplifies calculations where a symmetry is present.

So … pick a "Gaussian surface" that exploits the symmetry of the problem.

Problem 2.14. Find $\vec{E}(\vec{r})$ inside a sphere which carries charge with density proportional to the distance from the center: $\rho = kr$ (k is constant)

Choose as a Gaussian Surface a spherical shell of radius r. $\left(\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}\right)$

$$E\left(4\pi r^2\right) = \frac{1}{\varepsilon_0}\int \rho\, d\tau$$

$$= \frac{1}{\varepsilon_0}\int (kr')\left(r'^2 \sin\theta\, dr'\, d\theta\, d\phi\right)$$

$$= \frac{4\pi k}{\varepsilon_0}\int_0^r dr'\, r'^3 = \frac{r^4}{4}\frac{4\pi k}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi r^2 \varepsilon_0}\pi k r^4 \hat{r} = \frac{kr^2}{4\varepsilon_0}\hat{r}$$

Curl of Electric Field

The following holds for any static charge distribution:

It is straightforward to show that $\oint \vec{E}\cdot d\vec{l} = 0$

Then, applying Stokes' theorem $\int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{a} = \oint \vec{E}\cdot d\vec{l} = 0$

$$\boxed{\vec{\nabla}\times\vec{E} = 0}$$

The electric field is a vector quantity whose curl is zero.

From the theory of vector fields:

If the curl of a vector function $(\vec{A})$ vanishes everywhere, then $\vec{A}$ can be written as **the gradient of a scalar function.**

$$\Rightarrow \vec{\nabla}\times\vec{A} = 0 \iff \vec{A} = \vec{\nabla} f$$

Electric Potential

Since $\vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} V$ (minus sign is just by convention)

Where $V(\vec{r}) \equiv \int_{r}^{O} \vec{E}\cdot d\vec{l}$ (V: scalar; O: reference point (arbitrary))

from which it follows

$$\boxed{V(\vec{b}) - V(\vec{a}) = -\int_a^b \vec{E}\cdot d\vec{l}}$$

The actual value of the potential at a point is arbitrary (and, therefore, has no physical significance).

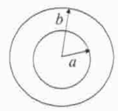
It is only the difference in potential that is physically meaningful.

By convention (and for convenience) we usually let reference point to be at ∞ and set $V(\infty) = 0$.

The potential obeys principle of superposition $\Rightarrow V = \sum V_i$

Problem 2.23. For the charge configuration:
hollow spherical shell with $\rho = \dfrac{k}{r^2}$ $\quad a \le r \le b$

Find V at the center: $V(0) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{l}$



Calculate $\vec{E}$ in 3 regions:

(*i*) $r < a \qquad Q_{encl} = 0 \quad \Rightarrow \quad \vec{E} = 0$

(*ii*) $a < r < b \qquad \oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}}{\varepsilon_0}$

$$E\left(4\pi r^2\right) = \frac{1}{\varepsilon_0}\int \rho\, d\tau = \frac{1}{\varepsilon_0}\int \frac{k}{r^2}\overbrace{r^2 \sin\theta\, dr\, d\theta\, d\phi}^{d\tau}$$

$$= \frac{4\pi k}{\varepsilon_0}\int_a^r dr = \frac{4\pi k}{\varepsilon_0}(r-a)$$

$$\vec{E} = \frac{k}{\varepsilon_0}\frac{(r-a)}{r^2}\hat{r}$$

(*iii*) $r > b \qquad E\left(4\pi r^2\right) = \dfrac{4\pi k}{\varepsilon_0}(b-a) \qquad \vec{E} = \dfrac{k}{\varepsilon_0}\dfrac{(b-a)}{r^2}\hat{r}$

$$\Rightarrow \quad V(0) = -\int_{\infty}^{b}\frac{k}{\varepsilon_0}\frac{(b-a)}{r^2}dr - \int_{b}^{a}\frac{k}{\varepsilon_0}\frac{(r-a)}{r^2}dr - \int_{a}^{0}(0)dr$$

$$= \frac{k}{\varepsilon_0}\frac{(b-a)}{b} - \frac{k}{\varepsilon_0}\left[\ln\left(\frac{a}{b}\right) + a\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$= \frac{k}{\varepsilon_0}\left[1 - \frac{a}{b} - \ln\left(\frac{a}{b}\right) - 1 + \frac{a}{b}\right]$$

$$= -\frac{k}{\varepsilon_0}\ln\left(\frac{a}{b}\right) = \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

2.3.3. Poisson's Equation and Laplace's Equation

Recall $\vec{E} = -\vec{\nabla}V$ and $\underbrace{\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}}_{\text{Gauss Law in differential form}} \quad \Rightarrow \quad \vec{\nabla}\cdot\left(-\vec{\nabla}V\right) = \dfrac{\rho}{\varepsilon_0}$

Poisson's Equation

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

Laplacian Operator

The special case of Poission Equation $\boxed{\rho = 0}$ is known as

Laplace's Equation

$$\boxed{\nabla^2 V = 0}$$

2.3.4. Localized Charge Distribution

For a **point charge**: $\boxed{V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\Re}} \qquad \Re = \left|\vec{r} - \vec{r}_q\right|$

For **many charges**:

discrete $\quad \boxed{V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{i=1}^{n}\frac{q_i}{\Re_i}}$

continuous $\quad \boxed{V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int\frac{dq}{\Re}}$

where dq could be a line, surface, or volume charge.

Example: volume charge density ρ

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{\Re}d\tau'$$

volume integral

2.4.1. The Work Done to move a Charge

Energy conservations

How much work is required to move charges around?

Recall: $W = \int_{\vec{a}}^{\vec{b}}\vec{F}\cdot d\vec{l} = \int_{\vec{a}}^{\vec{b}}\left(-Q\vec{E}\right)\cdot d\vec{l}$

force **you** exert to move charge

$$\left(\int_a^b \vec{F}\cdot d\vec{l} = \int_{\infty}^{b}\vec{F}\cdot d\vec{l} - \int_{\infty}^{a}\vec{F}\cdot d\vec{l} = V(\vec{b}) - V(\vec{a})\right)$$

$$= Q\left[-\int_{\vec{a}}^{\vec{b}}\vec{E}\cdot d\vec{l}\right]$$

$$\boxed{W = Q\left[V(\vec{b}) - V(\vec{a})\right]}$$

This is the work **you** must do to move a charge Q from point a to point b.

2.4.2. The Energy of a Point Charge Distribution

Potential Energy

Don't confuse "potential" and "potential energy"!!!

How much work is required to produce a configuration of charges?

Begin with all charges at ∞ :

- Bring in charge q_1 - no work.
- Bring in charge $q_2 \quad W_2 = q_2\left[V(\vec{r}_2) - V(\infty)\right] = q_2\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1}{\Re_{12}}$
- Bring in charge $q_3 \quad W_3 = q_3\left[\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1}{\Re_{13}} + \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_2}{\Re_{23}}\right]$

And so on.

The total work is then the sum of W for all charge pairs: $W = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{i=1}^{n}\sum_{\substack{j\\ j>i}}^{n}\frac{q_i q_j}{\Re_{ij}}$

or $\quad \boxed{W = \dfrac{1}{2}\dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{\Re_{ij}}}$

Or, in terms of potential

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

Potential energy of the system for discrete charges.

Potential at $\vec{r}_i$ (location at q_i) due to all other charges.

2.4.3. The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\int \rho V d\tau \qquad \begin{pmatrix}\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}\end{pmatrix} \quad \begin{pmatrix}\vec{E} = -\vec{\nabla}V\end{pmatrix}$$

But the potential is related (through its gradient) to the electric field.

We can therefore show

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$$

Energy stored in Electric Field.

Problem 2.34. Consider two concentric spherical shells of radii a and b.
Inner shell carries charge q
Outer shell carries charge $-q$

Calculate the energy of this configuration. $\left(W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau\right)$

1st approach:

Accounting for the Gauss' law, define the limits of integration:

$$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad (a < r < b)$$
$$\left|\vec{E}\right| = 0 \quad (elsewhere)$$

2nd approach:

Consider electric field as a sum of individual fields: $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$E^2 = \left(\vec{E}_1 + \vec{E}_2\right)^2 = E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2$$

$$\left|\frac{\varepsilon_0}{q}\int_{all\ space}(\quad)4\pi r^2 dr \quad \Rightarrow \quad W = W_1 + W_2 + \varepsilon_0\int \vec{E}_1\cdot\vec{E}_2 d\tau\right.$$

1st approach: $\left(W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau\right)$

$$W = \frac{\varepsilon_0}{2}\int\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 \overbrace{r^2 dr \sin\theta d\theta d\phi}^{d\tau}$$

$$W = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_{-\pi}^{\pi}\sin\theta d\theta \int_a^b \left(\frac{1}{r^2}\right)^2 r^2 dr$$

$$\Rightarrow \quad W = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 (2\pi)(2)\int_a^b \frac{1}{r^2}dr$$

from integration over ϕ — from integration over θ

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

2nd approach: $\left(W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau\right)$

$$W_{tot} = W_1 + W_2 + \varepsilon_0\int \vec{E}_1\cdot\vec{E}_2 d\tau$$

$$W_1 = \frac{\varepsilon_0}{2}\int E_1^2 d\tau = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\int_a^{\infty}\left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{a}$$

$$W_2 = \frac{\varepsilon_0}{2}\int E_2^2 d\tau = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\int_b^{\infty}\left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{b}$$

$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ $\quad r > a$ Field due to inner sphere

$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}\hat{r}$ $\quad r > b$ Field due to outer spherical shell

$$\Rightarrow \quad \vec{E}_1\cdot\vec{E}_2 = \left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{-q^2}{r^4} \qquad (r > b)$$

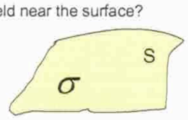
$$\varepsilon_0\int \vec{E}_1\cdot\vec{E}_2 d\tau = -\left(\frac{1}{4\pi\varepsilon_0}\right)^2 q^2\varepsilon_0\int_b^{\infty}4\pi r^2\frac{1}{r^4}dr = -\frac{q^2}{4\pi\varepsilon_0 b}$$

$$W_{tot} = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right] = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

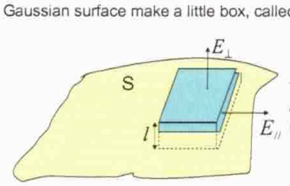
2.3.5. Electrostatic Boundary Conditions

Consider a surface which carries a surface charge density σ (charge/unit area).

What is the electric field near the surface?



Use the Gauss' Law to determine $\vec{E}$.
For a Gaussian surface make a little box, called a "Gaussian pillbox".



The "box top" (area A) is above S and the "box bottom" (area A) is below S.

Now apply Gauss' Law $\quad \oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$

Only the top and bottom of the box contribute to the integral (in the limit of an infinitesimal width box)

$$\Rightarrow \quad E_{above}^{\perp}A - E_{below}^{\perp}A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \quad E_{above}^{\perp} - E_{below}^{\perp} = \frac{\sigma}{\varepsilon_0}$$

There is a discontinuity in $E^{\perp}$ across a boundary containing a surface charge.

What about $E^{//}$? $\quad \oint \vec{E}\cdot d\vec{l} = 0$

$$E_{above}^{//}l - E_{below}^{//}l = 0$$

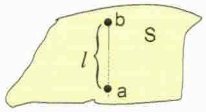
$$\Rightarrow \quad E_{above}^{//}l = E_{below}^{//}l$$

Combining $E^{\perp}$ and $E^{\parallel}$ components, we can write electric field in vector form

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Unit normal

What about potential? $V_{above} - V_{below} = -\int_a^b \vec{E}\cdot d\vec{l}$



But in the limit of $l\left(=\int dl\right)\to 0$

The integral vanishes: $V_{above} - V_{below} = 0$

$$V_{above} = V_{below}$$

But $\vec{E} = -\vec{\nabla}V$ $\left(\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}\right)$

$$-\vec{\nabla}V_{above} - \left(-\vec{\nabla}V_{below}\right) = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\vec{\nabla}V_{above} - \vec{\nabla}V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$$

Use normal derivative of V

$$\left(\begin{array}{l}\vec{\nabla}V = \frac{\partial V}{\partial n}\hat{n}\\ \vec{\nabla}V\cdot\hat{n} = \frac{\partial V}{\partial n}\hat{n}\cdot\hat{n}\\ \vec{\nabla}V\cdot\hat{n} = \frac{\partial V}{\partial n}\end{array}\right)$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

2.5.1. Conductors

Basic properties of a conductor:

(I) $\vec{E} = 0$ *inside*

(II) $\rho = 0$ *inside* because $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = 0$ if $\vec{E} = 0$

(III) net charge is on surface

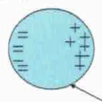
(IV) a conductor is an equipotential

(V) $\vec{E}$ is $\perp$ to the surface, just outside

2.5.2. Induced Charges

If you put a charge q next to a conductor it will **induce** charge:

$+q$



Cancel electric field inside

2.5.3. Surface Charge and the Force on a Conductor

Recall: $\left(\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}\right)$

For a conductor, $\vec{E}_{inside} = 0$, so $E_{below} = 0$

$$\Rightarrow \vec{E} = \vec{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

And, if E_{below}=0, then ∇V_{below}=0 (its an equipotential). So,

$$\vec{\nabla}V_{above} = -\frac{\sigma}{\varepsilon_0}\hat{n} \quad or \quad \frac{\partial V_{above}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

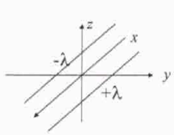
or rearranging $\sigma = -\varepsilon_0\frac{\partial V}{\partial n}$

Problem 2.47.

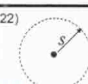
Consider the following charge configuration: Two infinitely long wires running parallel to the x-axis with uniform charge density $+\lambda$ and $-\lambda$.

(a) Find the potential at any point (x,y,z), using the origin as reference.

(b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential V_0.



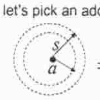
(a) Consider a single wire first (see problem 2.22)



$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$E\cdot 2\pi s\cdot l = \frac{\lambda l}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0\cdot s}\hat{s}$$

We cannot set reference point at infinity because charge extends to ∞.
So, let's pick an additional reference point at radius a.

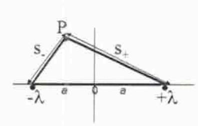


$$\Rightarrow V(s) = -\int_a^s E\cdot dl = -\int_a^s \frac{\lambda}{2\pi\varepsilon_0\cdot s}ds = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s}{a}\right)$$

Now extend result to two wires:

$$V_- = -\frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{a}\right)$$

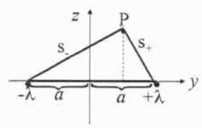
$$V_+ = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_+}{a}\right)$$



Hence, the total potential at some point P is

$$V = V_- + V_+ = \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\left(\frac{s_-}{a}\right) - \ln\left(\frac{s_+}{a}\right)\right] = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{s_+}\right)$$

In Cartesian coordinates:



$$s_+ = \left[(y-a)^2 + z^2\right]^{1/2}$$

$$s_- = \left[(y+a)^2 + z^2\right]^{1/2}$$

$$V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right]^{1/2}\right)$$

(b) So, what is the equipotential surface at $V = V_0$?

$$\left(\ln x^{\frac{1}{2}} = \frac{1}{2}\ln x\right)$$

$$\left(V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right]\right)$$

$$\frac{4\pi\varepsilon_0 V_0}{\lambda} = \ln\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right]$$

$$e^{4\pi\varepsilon_0 V_0/\lambda} = \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2} = const = k \qquad k = e^{4\pi\varepsilon_0 V_0/\lambda}$$

$$(y+a)^2 + z^2 = k\left[(y-a)^2 + z^2\right]$$

$$y^2 + 2ay + a^2 + z^2 - ky^2 + 2kay - ka^2 - kz^2 = 0 \quad \}\times(-1)$$

$$y^2(k-1) + z^2(k-1) + a^2(k-1) - 2ay(k+1) = 0 \quad \}:(k-1)$$

$$y^2 + z^2 + a^2 - 2ay\left(\frac{k+1}{k-1}\right) = 0$$

$$y^2 + z^2 - 2ay\left(\frac{k+1}{k-1}\right) + a^2 + a^2\left(\frac{k+1}{k-1}\right)^2 - a^2\left(\frac{k+1}{k-1}\right)^2 = 0$$

Let $y_0 = a\left(\frac{k+1}{k-1}\right)$

$$(y - y_0)^2 + z^2 - \underbrace{a^2\left(\frac{(k+1)^2}{(k-1)^2} - 1\right)}_{let\ it\ be\ R^2} = 0$$

$$R^2 = a^2\frac{(k+1+k-1)(k+1-k+1)}{(k-1)^2} = a^2\frac{2k\cdot 2}{(k-1)^2}$$

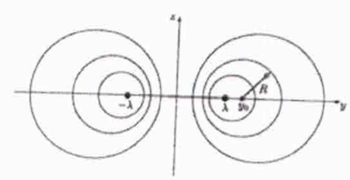
$$R = \frac{2a\sqrt{k}}{|k-1|}$$

Then we can write $y^2 + z^2 + a^2 - 2ay\left(\frac{k+1}{k-1}\right) = 0$ as:

$$(y - y_0)^2 + z^2 = R^2$$

This is equation of a circle.

Therefore, for two infinite line charges, the equipotential surfaces are (circular) cylinders of radius R offset from the origin by y_0.



Let us express y_0 and R in terms of V_0, λ, a $\qquad (V_0 > 0)$

$$y_0 = a\frac{k+1}{k-1} = a\frac{e^{4\pi\varepsilon_0 V_0/\lambda} + 1}{e^{4\pi\varepsilon_0 V_0/\lambda} - 1} = a\coth\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)$$

$$y_0 = a\frac{\cosh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}{\sinh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}$$

Similarly,

$$R = \frac{2a\sqrt{k}}{|k-1|} = 2a\frac{e^{2\pi\varepsilon_0 V_0/\lambda}}{e^{4\pi\varepsilon_0 V_0/\lambda} - 1} = a\frac{2}{\left(e^{2\pi\varepsilon_0 V_0/\lambda} - e^{-2\pi\varepsilon_0 V_0/\lambda}\right)}$$

$$R = \frac{a}{\sinh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}$$

PROBLEM 2.21

Problem 2.21 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

IDENTIFY Relevant concepts

By definition $$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}\cdot d\vec{l} \qquad \text{(Eq. 2.21)}$$

Reference point in infinity.

From problem 2.8:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$ for $r>R$ (<u>outside</u> the sphere)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r}$$ for $r<R$ (<u>inside</u> the sphere)

PROBLEM 2.21(cont.)

EXECUTE

For $r > R$:

$$V(r) = -\int_{\infty}^{r} E\cdot dr = -\int_{\infty}^{r}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)_{\infty}^{r} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}-0\right) = \boxed{\frac{q}{4\pi\varepsilon_0}\frac{1}{r}}$$

For $r < R$:

$$V(r) = -\int_{\infty}^{r} E\cdot dr = -\underbrace{\int_{\infty}^{R} E\cdot dr}_{outside} - \underbrace{\int_{R}^{r} E\cdot dr}_{inside} =$$

$$= -\int_{\infty}^{R}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)dr - \int_{R}^{r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}r\right)dr =$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)_{\infty}^{R} - \frac{q}{4\pi\varepsilon_0 R^3}\left(\frac{r^2}{2}\right)_{R}^{r} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - 0 - \frac{1}{R^3}\frac{r^2-R^2}{2}\right) =$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2R}\left(2 - \frac{r^2}{R^2} + 1\right) = \boxed{\frac{q}{4\pi\varepsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)}$$

PROBLEM 2.21(cont.)

EVALUATE

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}}_{=0 \text{ because of spherical symmetry}}$$

<u>When $r > R$</u>

$$\vec{\nabla}V = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} \Rightarrow \vec{E} = -\vec{\nabla}V = \boxed{\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}}$$

<u>When $r < R$</u>

$$\vec{\nabla}V = \frac{q}{4\pi\varepsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{r} = \frac{q}{4\pi\varepsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{r} =$$

$$\vec{\nabla}V = -\frac{q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{r} \Rightarrow \vec{E} = -\vec{\nabla}V = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}r\hat{r}}$$

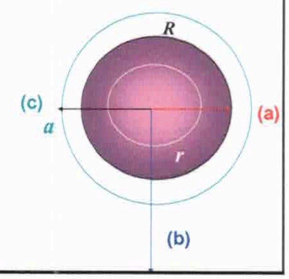
PROBLEM 2.32

Problem 2.32 Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do it three different ways:

(a) Use Eq. 2.43. You found the potential in Prob. 2.21.

(b) Use Eq. 2.45. Don't forget to integrate over *all space*.

(c) Use Eq. 2.44. Take a spherical volume of radius a. What happens as $a \to \infty$?



PROBLEM 2.32

IDENTIFY Relevant concepts

(a) (Eq. 2.43)

$$W = \frac{1}{2}\int_{\substack{Sphere\\volume}} \rho V(r,\theta,\varphi)\,d\tau$$

ρ - charge density (uniform)

V – potential <u>**inside**</u> the sphere (in spherical coordinates)

$d\tau$ – element of the volume

From Problem 2.21 ($r < R$):

$$V(r,\theta,\varphi) = V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Due to symmetry, spherical shell of radius r and thickness dr can be used:

$$d\tau = 4\pi r^2 dr$$

PROBLEM 2.32 (cont.)

IDENTIFY Relevant Concepts (cont.)

(b) (Eq. 2.45) $$W = \frac{\varepsilon_0}{2}\int_{\substack{all\\space}} E^2(r,\theta,\varphi)\,d\tau$$

Spherical symmetry

From problem 2.8:

$$\vec{E} = \begin{cases} \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r} & \text{for } r > R \text{ (outside the sphere)} \\ \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qr}{R^3}\hat{r} & \text{for } r < R \text{ (inside the sphere)} \end{cases}$$

(c) (Eq. 2.44) $$W = \frac{\varepsilon_0}{2}\left(\int_{\substack{enclosed\\volume}} E^2 d\tau + \int_{surface} V\vec{E}\cdot d\vec{a}\right)$$

- Enclosed volume or enclosing surface: $a>R$
- At the end we will let $a\to\infty$

PROBLEM 2.32 (cont.)

SET UP & EXECUTE

(a)

$$W=\frac{1}{2}\int_0^R \rho\frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3-\frac{r^2}{R^2}\right)4\pi r^2dr=\frac{1}{2}\rho\frac{1}{4\pi\varepsilon_0}\frac{q}{2R}4\pi\int_0^R\left(3-\frac{r^2}{R^2}\right)r^2dr$$

$$=\frac{q\rho}{4\varepsilon_0 R}\left[3\frac{r^3}{3}-\frac{1}{R^2}\frac{r^5}{5}\right]_0^R=\frac{q\rho}{4\varepsilon_0 R}\left[R^3-\frac{R^3}{5}\right]=\frac{q\rho R^2}{5\varepsilon_0}$$

$$\rho=\frac{q}{\frac{4}{3}\pi R^3}=\frac{3q}{4\pi}\frac{1}{R^3}$$

$$\boxed{W=\frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{q^2}{R}}$$

PROBLEM 2.32 (cont.)

EXECUTE (cont.)

(b)

$$W=\frac{\varepsilon_0}{2}\left\{\int_0^R E^2_{inside}\overbrace{4\pi r^2dr}^{d\tau}+\int_R^\infty E^2_{outside}\overbrace{4\pi r^2dr}^{d\tau}\right\}=$$

$$=\frac{\varepsilon_0}{2}\frac{1}{(4\pi\varepsilon_0)^2}q^2 4\pi\left\{\int_0^R\left(\frac{r}{R^3}\right)^2r^2dr+\int_R^\infty\frac{1}{r^4}r^2dr\right\}=$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{R^6}\left(\frac{r^5}{5}\right)_0^R+\left(-\frac{1}{r}\right)_R^\infty\right\}=$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{5R}+\frac{1}{R}\right\}=\boxed{\frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{q^2}{R}}$$

PROBLEM 2.32 (cont.)

EXECUTE (cont.)

(c)

$$W=\frac{\varepsilon_0}{2}\left(\int_{\substack{enclosed\\volume\\r=a}}E^2d\tau+\int_{\substack{surface\ at\\r=a}}V\vec{E}\cdot d\vec{a}\right)=$$

$$=\frac{\varepsilon_0}{2}\left(\int_0^R E^2_{inside}4\pi r^2dr+\int_R^a E^2_{outside}4\pi r^2dr+\int_{r=a}V_{outside}E_{outside}\overbrace{r^2\sin\theta d\theta d\phi}^{da}\right)$$

$$=\frac{\varepsilon_0}{2}\left\{\frac{1}{(4\pi\varepsilon_0)^2}\frac{q^2}{R^6}4\pi\int_0^R r^4dr+\frac{1}{(4\pi\varepsilon_0)^2}q^2 4\pi\int_R^a\frac{1}{r^4}r^2dr+\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a}\right)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\right)a^2\overbrace{\int_{-\pi}^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi}^{4\pi}\right\}$$

$$=\frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 4\pi\left\{\frac{1}{5R}+\frac{1}{R}-\frac{1}{a}+\frac{1}{a}\right\}=\boxed{\frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{q^2}{R}}$$

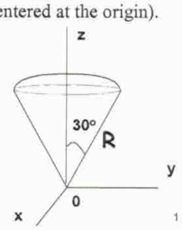
EVALUATE

PROBLEM 1.58

Check the divergence theorem for the function

$$\vec{V} = r^2 \sin\theta \hat{r} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \hat{\phi},$$

using the volume of the cone shown in the figure (the top surface is spherical, with radius R and centered at the origin).

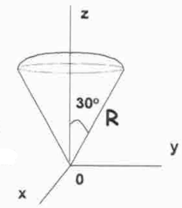


PROBLEM 1.58

IDENTIFY Relevant concepts

(a) Green's (divergence) theorem (Eq. 1.56):

$$\underbrace{\int}_{\text{all enclosed volume } v}(\vec{\nabla}\cdot\vec{V})\underbrace{d\tau}_{\text{element of the enclosed volume}} = \underbrace{\oint}_{\text{surface } S} \vec{V}\cdot \underbrace{d\vec{a}}_{\text{element of the surface}}$$



(b) Volume ("ice cream cone") is a conical section of the sphere with surface composed of a "cone" and an "ice cream".

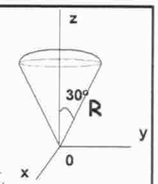
(c) Geometry of the problem and function formulation suggest spherical coordinate system.

PROBLEM 1.58

SET UP

$$\vec{V} = V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi},$$

$$V_r = r^2\sin\theta \quad V_\theta = 4r^2\cos\theta \quad V_\phi = r^2\tan\theta$$



$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2r^2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta) =$$

$$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2(\cos^2\theta - \sin^2\theta) + 0 =$$

$$= \frac{4r}{\sin\theta}(\sin^2\theta + \cos^2\theta - \sin^2\theta) =$$

$$= 4r\frac{\cos^2\theta}{\sin\theta}$$

PROBLEM 1.58

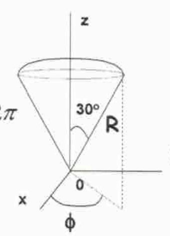
SET UP

Defining the enclosed volume in terms of spherical coordinates:

$$r: 0 \to R; \quad \theta: 0 \to \pi/6; \quad \phi: 0 \to 2\pi$$

Volume element

$$d\tau = r^2\sin\theta \, dr \, d\theta \, d\phi$$



EXECUTE

$$\int_{volume}(\vec{\nabla}\cdot\vec{V})d\tau = \int_{volume}\left(4r\frac{\cos^2\theta}{\sin\theta}\right)(r^2\sin\theta dr d\theta d\phi) =$$

$$= \int_0^R 4r^3 dr \int_0^{\pi/6}\cos^2\theta d\theta \int_0^{2\pi} d\phi = I_1 \times I_2 \times I_3$$

PROBLEM 1.58

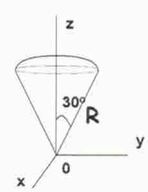
EXECUTE

$$I_1 = \int_0^R 4r^3 dr = R^4$$

$$I_2 = \int_0^{\pi/6}\cos^2\theta d\theta = \int_0^{\pi/6}\frac{1+\cos\left(\overset{\Theta}{2\theta}\right)}{2}d\theta$$

$$I_2 = \frac{1}{2}\int_0^{\pi/6}d\theta + \frac{1}{4}\int_0^{\pi/3}\cos(\Theta)d\Theta = \frac{\pi}{12} + \frac{\sin\left(\frac{\pi}{3}\right)}{4} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{1}{24}(2\pi + 3\sqrt{3})$$

$$I_3 = \int_0^{2\pi} d\phi = 2\pi$$



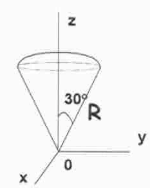
$$\int_{volume}(\vec{\nabla}\cdot\vec{V})d\tau = R^4(2\pi)\frac{1}{24}(2\pi + 3\sqrt{3}) = \boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$$

PROBLEM 1.58

SET UP & EVALUATE

We check the validity of divergence theorem by showing that the right-side integral gives the same value as the left-side integral.

$$\underbrace{\int}_{\text{all enclosed volume } v}(\vec{\nabla}\cdot\vec{V})\underbrace{d\tau}_{\text{element of the enclosed volume}} = \underbrace{\oint}_{\text{surface } S} \vec{V}\cdot \underbrace{d\vec{a}}_{\text{element of the surface}}$$



Enclosing surface can be split into two surfaces, (a) the "ice cream" and (b) the "cone"

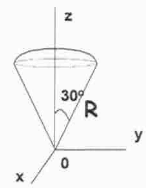
We can define these two partially enclosing surfaces in terms of spherical coordinates as follows:

The "ice cream" $\quad r = R \quad \theta: 0 \to \pi/6 \quad \phi: 0 \to 2\pi$

The "cone" $\quad r: 0 \to R \quad \theta = \pi/6 \quad \phi: 0 \to 2\pi$

PROBLEM 1.58

SET UP & EVALUATE



Note that the surface elements of two surfaces are different:

The "ice cream"

$$d\vec{a} = R^2 \sin\theta d\theta d\phi \hat{r} = da\hat{r}$$

$$\vec{V}\cdot d\vec{a} = V_r da = (R^2 \sin\theta)(R^2 \sin\theta d\theta d\phi) =$$

$$= R^4 \sin^2\theta d\theta d\phi$$

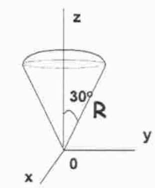
The "cone"

$$d\vec{a} = r\sin\theta dr d\phi \hat{\theta} = da_\theta \hat{\theta}$$

$$\vec{V}\cdot d\vec{a} = V_\theta da_\theta = (4r^2\cos\theta)(r\sin\theta dr d\phi) = \sqrt{3}\, r^3 dr d\theta$$

PROBLEM 1.58

EVALUATE



$$\oint_{\text{surface } S} \vec{V}\cdot \underbrace{d\vec{a}}_{\text{element of the surface}} =$$

$$= \int_{\text{"icecream"}} \vec{V}\cdot d\vec{a} + \int_{\text{"cone"}} \vec{V}\cdot d\vec{a} = I_1 + I_2$$

$$I_1 = R^4 \int_0^{\pi/6} \sin^2\theta d\theta \int_0^{2\pi} d\phi = 2\pi R^4 \int_0^{\pi/6} \frac{1-\cos(2\theta)}{2} d\theta =$$

$$2\pi R^4\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$$

$$I_2 = \sqrt{3}\int_0^R r^3 dr \int_0^{2\pi} d\phi = \frac{\pi R^4}{6} 3\sqrt{3}$$

$$\oint_S \vec{V}\cdot d\vec{a} = I_1 + I_2$$

$$= \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)$$

Problem 2.2

(a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges, q, a distance d apart (Fig. 2.4). Check that your result is consistent with what you'd expect when $z \gg d$.

(b) Repeat part (a), only this time make the right-hand charge $-q$ instead of $+q$.

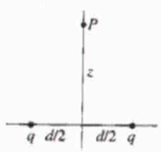


Figure 2.4

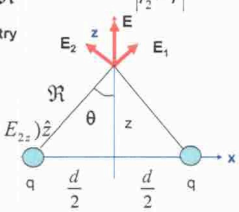
PROBLEM 2.2a

IDENTIFY

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re^2}\hat{\Re} \qquad \hat{\Re} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}|}$$

Superposition of vector $\vec{E}$ Symmetry

SET UP Draw a diagram



EXECUTE $\vec{E} = (E_{1x} + E_{2x})\hat{x} + (E_{1z} + E_{2z})\hat{z}$

Symmetry: $E_{1x} = -E_{2x} \therefore E_x = 0$

$E_{1z} = E_{2z} \therefore E_z = 2E_{1z}$

$$E_{1z} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re^2}\cos\theta \qquad \Re^2 = z^2 + \left(\frac{d}{2}\right)^2; \cos\theta = \frac{z}{\Re} = \frac{z}{\sqrt{z^2 + \left(\frac{d}{2}\right)^2}}$$

$$\vec{E} = E_z\hat{z} = 2E_{1z}\hat{z} = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)}\frac{z}{\sqrt{z^2 + \left(\frac{d}{2}\right)^2}}\hat{z}$$

PROBLEM 2.2a (cont.)

EXECUTE (cont.)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}}\hat{z}$$

EVALUATE

$z >> d \equiv d \to 0$

$$\Rightarrow \left(\lim \vec{E}\right)_{d\to 0} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left(z^2\right)^{\frac{3}{2}}}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{z^3}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{z^2}\hat{z}$$

Far away, two charges look as a single charge **2q**

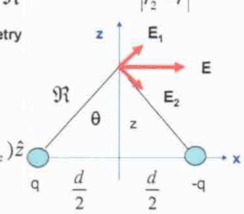
PROBLEM 2.2b

IDENTIFY

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re^2}\hat{\Re} \qquad \hat{\Re} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}|}$$

Superposition of vector $\vec{E}$ Symmetry

SET UP Draw a diagram



EXECUTE Superposition:

$$\vec{E} = (E_{1x} + E_{2x})\hat{x} + (E_{1z} + E_{2z})\hat{z}$$

Symmetry: $E_{1z} = -E_{2z} \therefore E_z = 0$

$E_{1x} = E_{2x} \therefore E_x = 2E_{1x}$

$$E_{1x} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re^2}\sin\theta \qquad \Re^2 = z^2 + \left(\frac{d}{2}\right)^2; \sin\theta = \frac{d/2}{\Re} = \frac{d}{2\sqrt{z^2 + \left(\frac{d}{2}\right)^2}}$$

PROBLEM 2.2b (cont.) EXECUTE (cont.)

$$\vec{E} = E_x\hat{x} = 2E_{1x}\hat{x} = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(z^2+\left(\frac{d}{2}\right)^2\right)}\frac{d}{2\sqrt{z^2+\left(\frac{d}{2}\right)^2}}\hat{x}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}}\hat{x}$$

EVALUATE

$$x,z >> d \Rightarrow \vec{E} \approx \frac{1}{4\pi\varepsilon_0}\frac{qd}{z^3}\hat{x}$$

$$d \to 0 \Rightarrow \vec{E} = 0.$$

From far away, the field looks like a the field of a dipole with dipole moment **qd** 13

Problem 2.10 A charge q sits at the back corner of a cube, as shown in Fig. 2.17. What is the flux of **E** through the shaded side?

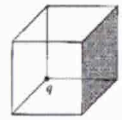


Figure 2.17

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PROBLEM 2.10 IDENTIFY Flux of the electric field $\vec{E}$ Through a surface **S** $\Phi_E = \int_S \vec{E}\cdot d\vec{a}$

"Gauss' Law" $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$

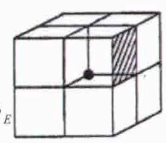
Symmetry

SET UP **Draw a diagram (a larger cube where q is at center)**

EXECUTE

$$\Phi_{total} = \oint_{larger\ cube}(\vec{E}d\vec{a}) = \frac{q}{\varepsilon_0}$$

Symmetry: $\Phi_{total} = (4\times 6)\Phi_{side\ of\ smaller\ cube} = 24\Phi_E$



$$\Phi_E = \frac{q}{24\varepsilon_0}$$

EVALUATE 15

Problem 2.16 A long coaxial cable (Fig. 2.26) carries a uniform *volume* charge density ρ on the inner cylinder (radius a), and a uniform *surface* charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: (i) inside the inner cylinder ($s < a$), (ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of s.

Figure 2.26

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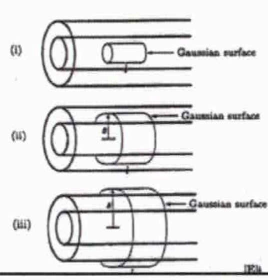
PROBLEM 2.16 IDENTIFY Relevant concepts **"Gauss' Law"** $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$

Cylindrical symmetry; volume and area of a cylinder

SET UP **Draw diagrams with Gaussian surfaces**



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PROBLEM 2.16 (cont.) EXECUTE

$$(i)\ldots \oint_{Gaussian\ surface}\vec{E}\bullet d\vec{a} = E\cdot 2\pi s\cdot l = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho V_{Gaussian\ volume}}{\varepsilon_0} = \frac{\rho\pi s^2 l}{\varepsilon_0}$$

$$\ldots\ldots E = \frac{\rho s}{2\varepsilon_0}\hat{s}$$

$$(ii)\ldots \oint_{Gaussian\ surface}\vec{E}\bullet d\vec{a} = E\cdot 2\pi s\cdot l = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho V_{Charge\ volume}}{\varepsilon_0} = \frac{\rho\pi a^2 l}{\varepsilon_0}$$

$$\ldots\ldots E = \frac{\rho a^2}{2\varepsilon_0 s}\hat{s}$$

$$(iii)\ldots \oint_{Gaussian\ surface}\vec{E}\bullet d\vec{a} = E\cdot 2\pi s\cdot l = \frac{Q_{encl}}{\varepsilon_0} = \frac{Q_{volume}-Q_{surface}}{\varepsilon_0} = 0 \therefore E = 0$$

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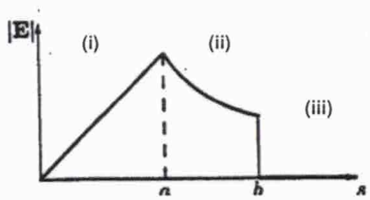
PROBLEM 2.16 (cont.) EVALUATE



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Problem 2.18 Two spheres, each of radius R and carrying uniform charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (Fig. 2.28). Call the vector from the positive center to the negative center **d**. Show that the field in the region of overlap is constant, and find its value. [*Hint:* Use the answer to Prob. 2.12.]

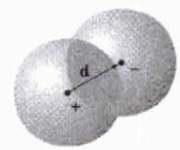

Figure 2.28

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PROBLEM 2.18

IDENTIFY Relevant concepts

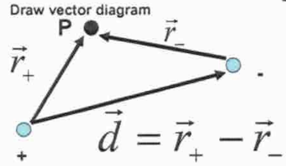
From Problem 2.12 electric field inside a sphere with uniform charge density ρ is proportional to the radial distance from center:

Superposition

$$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r} = \frac{\rho}{3\varepsilon_0}\vec{r}$$

SET UP

Draw vector diagram



P is a point in the overlapping part

$$\vec{d} = \vec{r}_+ - \vec{r}_-$$

EXECUTE

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\varepsilon_0}\vec{d}$$

EVALUATE

$$\vec{d} = const \Rightarrow \vec{E} = const$$

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PROBLEM 2.12

Problem 2.12 Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ). Compare your answer to Prob. 2.8.

IDENTIFY

"Gauss' Law"

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$$

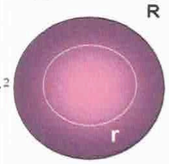
Spherical symmetry: $\vec{E} \parallel \vec{a}$ at every point on the surface

SET UP & EXECUTE

Gaussian surface: Sphere with radius r < R

use spherical symmetry $\oint_{\text{Gaussian surface}} \vec{E}\cdot d\vec{a} = E\cdot 4\pi r^2$

uniform charge density $Q_{encl} = \frac{4}{3}\pi r^3\rho$



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PROBLEM 2.12 (cont.)

EXECUTE (cont.)

Apply Gauss law $$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3\rho$$

Solve for E (direction is always radial due to spherical symmetry):

$$\vec{E} = \frac{1}{3\varepsilon_0}\rho r\hat{r} = \frac{\rho}{3\varepsilon_0}\vec{r}$$

EVALUATE

$$Q_{tot} = V_{sphere}\rho = \frac{4}{3}\pi R^3\rho \Rightarrow \rho = \frac{3Q_{tot}}{4\pi R^3}$$

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r} = \frac{\frac{3Q}{4\pi R^3}}{3\varepsilon_0}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}\vec{r}$$ as in Problem 2.8

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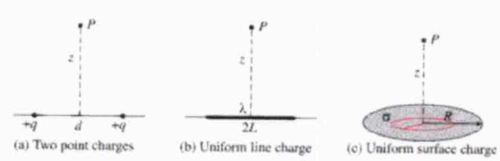
PROBLEM 2.25

Problem 2.25 Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Prob. 2.2a, Ex. 2.1, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2b, and explain carefully any discrepancy.



(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

Figure 2.34

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PROBLEM 2.25

IDENTIFY Relevant concepts

(a)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_1^2 \frac{q_i}{\Re_i} \qquad \text{(Eq. 2.27)}$$

(b)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\Re}dl' \qquad \text{(Eq. 2.30)}$$

(c)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\Re}da' \qquad \text{(Eq. 2.30)}$$

Use Tables of Integrals!

SET UP & EXECUTE

(a)

$$V = \frac{1}{4\pi\varepsilon_0}\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}$$

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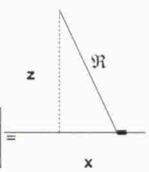
PROBLEM 2.25 (cont.)

(b) $\Re = \sqrt{z^2+x^2}$

SET UP & EXECUTE (cont.)



$$V = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda dx}{\sqrt{z^2+x^2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(x+\sqrt{z^2+x^2}\right)\right]_{-L}^{L} =$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\left(L+\sqrt{z^2+L^2}\right)^2}{z^2+L^2-L^2}\right] =$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{z}\right]$$

(c)

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{\sigma 2\pi r dr}{\sqrt{r^2+z^2}} = \frac{1}{4\pi\varepsilon_0}2\pi\sigma\left(\sqrt{r^2+z^2}\right)_0^R = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+z^2}-z\right)$$

EVALUATE

By symmetry $\frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} = 0 \Rightarrow \vec{E} = -\frac{\partial V}{\partial z}\hat{z}$

PROBLEM 2.25 (cont.) **EVALUATE (cont.)**

(a)

$$\vec{E} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}\right)\hat{z} = -\frac{2q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\frac{2z}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}}\hat{z} =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}}\hat{z}$$

See Problem 2.2a

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