(Chapter 3) Special Techniques

This Chapter address the practical solutions of the primary task of electrostatics:
Given charge distribution $\rho(\vec{r})$, what is electric field distribution, $E(\vec{r})$?

According to Coulomb's Law $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2}\rho(\vec{r}')d\tau'$ Usually, it is difficult to calculate integral!

Somewhat easier: • to calculate first potential $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\rho(\vec{r}')d\tau'$

• to apply symmetry and use Gauss's Law

Easier way is to solve problem in differential form.

3.1. Laplace Equation

3.1.1. Introduction

Recall Poisson's Equation: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

We are looking for the region where $\rho = 0$.
This reduces Poisson's Equation to Laplace's Equation: $\nabla^2 V = 0$

Let's look to solutions of Laplace's Equation.

At first, we will work in Cartesian coordinates: $(3-D)$ $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$V(x,y,z)$ is a harmonic function.

3.1.2. Laplace Equation in One Dimension

$(1-D)$ $\frac{\partial^2 V}{\partial x^2} = 0$ $V(x) = mx + b$ (Constants given by boundary conditions)

One variable $\Longrightarrow$ can be written $\frac{d^2}{dx^2}$ (Ordinary differential equation)

Properties of solutions:

- $V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$ *for any a* Laplace's averaging equation
- No local maxima or minima (extremes).
  Extreme values of $V(x)$ occurs only at the boundaries.
  $\left(\frac{dV}{dx} \neq 0\right)$

3.1.3. Laplace Equation in Two Dimensions

$(2-D)$ $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ (Partial differential equation)

Properties of solutions:

- $V(x,y) = \frac{1}{2\pi R}\oint_{circle} V dl$ (V at a given point is the average of those around the point)
- V has no extrema (except on boundary) $\Longrightarrow$ **Earnshaw's Theorem**:
  $\left(\frac{\partial V}{\partial x} \neq 0, \frac{\partial V}{\partial y} \neq 0\right)$
  A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.

3.1.4. Laplace Equation in Three Dimensions

In (3-*D*) the solutions have the same properties as in (1-*D*) and (2-*D*):

- $V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{spherical\ shell} V da$ (Average over the spherical surface)
- no extrema except on boundary

Problem 3.3.

(a) Find the general solution to Laplace's equation in spherical coordinates for the case where V depends only on r, $V = V(\vec{r})$

(b) Do the same for cylindrical coordinates, assuming V depends only on s.

(a) In spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Given $V = V(\vec{r})$ $\Rightarrow$ $\frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial\phi} = 0$

$\Rightarrow$ $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$ (Ordinary differential equation)

$\Rightarrow$ $r^2\frac{dV}{dr} = c_1$ $c_1 = const.$

$\frac{dV}{dr} = \frac{c_1}{r^2}$ $\Rightarrow$ $V = -\frac{c_1}{r} + c_2$ $c_2 = const.$

(b) In cylindrical coordinates given, V depends only on s.

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$V(\vec{r}) = V(s)$ $\Rightarrow$ $\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial z} = 0$

$\Rightarrow$ $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$

$s\frac{\partial V}{\partial s} = c_1$

$\frac{\partial V}{\partial s} = \frac{c_1}{s}$ $V = c_1 \ln s + c_2$

3.1.5. Boundary Conditions and Uniqueness Theorem

It is not enough to formulate Laplace Equation.
(In Problen 3.3 constants c_1 and c_2 are not defined.)

To determine conditions which arise when solving Laplace's Equation, we must know the boundary conditions.

For example, in order to determine potential one must define boundary conditions.

First Uniqueness Theorem

The solution to Laplace's Equation in some volume V is **uniquely** determined if the potential V is specified on the boundary surface.

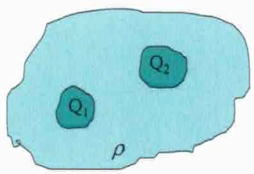
This means that, provided

- V is a solution to $\nabla^2 V = 0$
- V is correctly specified value on the boundary

$\Rightarrow$ V is specified everywhere inside the surface.

3.1.6. Conductors and Second Uniqueness Theorem

In a volume V surrounded by conductors and containing a spherical charge density ρ, the electric field is **uniquely** determined if the total charge on each conductor is given.

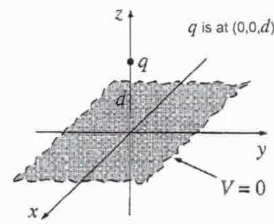


i.e. only one field configuration exists given Q_1, Q_2, ... ρ.

3.2. The Method of Images

3.2.1. The Classic Image Problem

Consider a charge q on the z-axis above a plane conductor (z=0) which is grounded (V=0).



What is V in the region above the plane?
It is not the same case as it would be if the conducting plane were absent.

We need to solve Poisson's Equation for z>0 with a single point charge, q. $\left(\nabla^2 V = -\frac{\rho}{\varepsilon_0}\right)$

Boundary conditions:

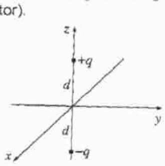
- V=0 for z=0
- $V \to 0$ for $x^2+y^2+z^2 >> d^2$

This problem has a unique solution.

Method of images: treat the conducting plane as a mirror and replace the conducting plane with an image charge.

The image charge has the same magnitude as the original charge, but opposite sign (for a plane conductor).

So for a charge **+*q*** at (0,0,d) the image is at (0,0,-d) and has a charge **−*q***.



Proof

Consider the two point charges and no conductive plane.

$$\Rightarrow\ V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left[x^2+y^2+(z-d)^2\right]^{1/2}}+\frac{-q}{\left[x^2+y^2+(z+d)^2\right]^{1/2}}\right]$$

At position of the plane: $z=0$ $\quad V=0$

Far away: $x^2+y^2+z^2>>d^2$ $\quad V=0$

V of the system of two mirror charges satisfies Laplace Equation and the same boundary conditions as the charge q at z-axis above the grounded infinite conducting plane at z=0.
According to the First Uniqueness theorem, these two systems (one charge and conducting plane and two charges) are equivalent.

3.2.2. Induced Surface Charge

- Recall the surface charge density $\sigma=-\varepsilon_0\frac{\partial V}{\partial n}$ (Section 2.5.3.)
- For grounded plane in Section 3.2.1. $\hat{n}=\hat{z}$

$$\sigma=-\varepsilon_0\frac{\partial V}{\partial z}\Big|_{z=0}$$

$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left[x^2+y^2+(z-d)^2\right]^{1/2}}-\frac{q}{\left[x^2+y^2+(z+d)^2\right]^{1/2}}\right]$$

$$\frac{\partial V}{\partial z}=\frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}-\frac{-q(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$\Rightarrow\ \sigma=\frac{-qd}{2\pi\left(x^2+y^2+d^2\right)^{3/2}}$$

Induced surface charge is negative as expected since the point charge is positive.

The largest value at x=y=0

and the **total induced charge** is $Q=\int_{x-y\ plane}\sigma da$

$$da=dxdy$$

In polar coordinates $x^2+y^2=r^2$ $(z=0)$

$$da=rdrd\phi$$

$$Q=\int_0^{\infty}\int_0^{2\pi}\frac{-qd}{2\pi\left(r^2+d^2\right)^{3/2}}rdrd\phi=\frac{qd}{\sqrt{r^2+d^2}}\Bigg|_0^{\infty}=-q$$

3.2.3. Force and Energy

So, the charge q induces a surface charge $\sigma = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$ on the plane.

We can use the potential we found in the upper half plane to calculate fields and force, but remember we are working only in the upper half plane. Force is, however, equivalent to the attractive force between charge q and its image $-q$: $\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$

However, the energy is $\frac{1}{2}$ what we would get from simply having two charges.

Why? $\vec{E}$ does not exist in the lower plane and $W \propto \int_{all\ space} E^2 d\tau$

Proof:

The energy is equal to the work done to bring the charge q from ∞ to $z=d$.

$\infty \rightarrow d$

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z)^2}\hat{z} \quad and \quad W = -\int_\infty^d \vec{F}\cdot\vec{l}$$

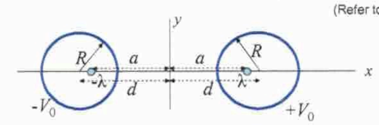
$$W = \frac{1}{4\pi\varepsilon_0}\int_\infty^d \frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_\infty^d = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

$$W = \frac{1}{2}\times \underbrace{\frac{q^2}{4\pi\varepsilon_0 2d}}_{\substack{\text{potential energy of two charges}\\ \left(q \text{ and its mirror charge } -q\right)}}$$

Problem 3.11. **Find V everywhere.**

Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart. One is at potential V_0, the other at $-V_0$.

(Refer to Problem 2.47)



The problem reduces to two long linear image charges $-\lambda$ and λ separated by $2a$. Following substitution hold:

Problem 2.47 → Problem 3.11

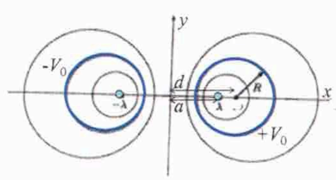
$y_0 \rightarrow d$

$R \rightarrow R$ .... $\Rightarrow$

$y \rightarrow x$

$z \rightarrow y$

As seen in Problem 2.47 these two image charges generate two cylindrical equipotential surfaces, one with $-V_0$ and other V_0.



From Problem 2.47:

$$d^2(=y_0^2) = a^2\frac{(k+1)^2}{(k-1)^2}$$

$$R^2 = a^2\frac{4k}{(k-1)^2}$$

$$d^2 - R^2 = a^2\left(\frac{(k+1)^2}{(k-1)^2} - \frac{4k}{(k-1)^2}\right) \qquad \boxed{a^2 = d^2 - R^2}$$

Potential everywhere is:

$$\boxed{V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]} \quad \text{with} \quad a^2 = d^2 - R^2$$

Also,

$$\frac{d}{R}\left(=\frac{y_0}{R}\right) = \frac{a\dfrac{\cosh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}{\sinh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}}{\dfrac{a}{\sinh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)}} = \cosh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)$$

$$\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right) = \cosh^{-1}\left(\frac{d}{R}\right)$$

$$\Rightarrow \quad \lambda = \frac{2\pi\varepsilon_0 V_0}{\cosh^{-1}\left(\frac{d}{R}\right)}$$

3.3. Separation of Variables

A general technique for solving differential equations is to use the method of Separation of variables.

Recall, in Modern Physics you did this for Schrödinger Equation.

Basic idea: Search for solutions which are products of functions that each depend on only one variable.

For example, solving Laplace's Equation $\nabla^2 V = 0$ by looking for solutions

$$V(x,y) = X(x)Y(y).$$

Subject, of course, to the appropriate boundary conditions.

Now the solution obtained may be an infinite sum, but that's OK provided the boundary conditions are satisfied.

Is this true for all sums?

NO! In order to satisfy general boundary conditions, the functions in the sum must form a "basis." The basis set is comprised of orthogonal functions.

Completeness of a set of functions

Let $f_n(y)$ form a basis on the interval $0 \le y \le a$.

A set of functions $f_n(y)$ is said to be complete if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

Orthogonality

A set of functions is orthogonal if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y) dy = 0 \qquad n' \neq n$$

What do the solutions look like?

It depends on the coordinate system used.

3.3.1. Cartesian Coordinates

Consider two dimensions in Cartesian coordinates.

Then Laplacian is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *and* $\nabla^2 V = 0$

Let's look for solution in the form of products $V(x,y) = X(x)Y(y)$

Then Laplace equation gets a form $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$

This will only be satisfied in general if: $\frac{1}{X}\frac{d^2X}{dy^2} = const. = C_1$

$\frac{1}{Y}\frac{d^2Y}{dy^2} = const. = C_2$

Subject to $C_1 + C_2 = 0$

C_1 may be positive or negative, and C_2 will then be opposite in sign.

How do you know the sign of C_1 and C_2?
⇒ Depends on boundary conditions.

For $C_1 < 0$ (call it $-k^2$)

$$\frac{d^2X}{dx^2} = -k^2 X \qquad \Rightarrow X(x) = A\sin kx + B\cos kx$$

then $C_2 = k^2 > 0$

$$\frac{d^2Y}{dy^2} = k^2 Y \qquad \Rightarrow Y(y) = Ce^{ky} + De^{-ky}$$

$$V(x,y) = X(x)Y(y) = (A\sin kx + B\cos kx)(Ce^{ky} + De^{-ky})$$

For $C_1 > 0$ (call it k^2)

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = C\sin ky + D\cos ky$$

$$V(x,y) = X(x)Y(y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

What about the constants A, B, C, D, k?

- Apply boundary conditions
- The boundary conditions will apply constraints on the constants.

Note: Since k can take on multiple values, the solution may have an infinite number of terms.

In general (for $C_1 > 0$)

$$V(x,y) = \sum_{allowed\ k} (A_k e^{kx} + B_k e^{-kx})(C_k \sin ky + D_k \cos ky)$$

Usually some of the constants ($A, B, C,$ or D) will be zero. The remaining constants can then be grouped together into a "new" constant.

3.3.2. Spherical Coordinates

In spherical coordinates, we follow a similar procedure as in Cartesian coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi^2}$$

For now we will only consider potentials which are independent of ϕ and look for solutions of the form:

$$V(r,\theta) = R(r)\Theta(\theta)$$

which leads to

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{C_1 = l(l+1)} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{C_2 = -l(l+1)} = 0$$

For convenience, we defined constants $C_1 = l(l+1)$ and $C_2 = -l(l+1)$.

It follows $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)R$ and for $\Theta(\theta)$ is more difficult to proof.

The general solution in spherical coordinates is then:

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$\Theta(\theta) = P_l(\cos\theta)$$

"Legendre polynomials"

Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2 - 1)/2 \\ P_3(x) &= (5x^3 - 3x)/2 \end{aligned}$$

....

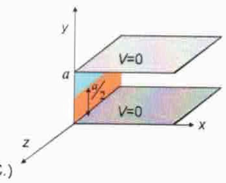
Therefore, the general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l[\cos\theta]$$

Problem 3.12. (Refer to Section 3.3.1.)

Find the potential in the infinite slot given in the Figure below if the boundary at x=0 consists of two metal strips, one from y=0 to y=a/2 is held at a constant potential V_0, and the other, from y=a/2 to y=a is at potential $-V_0$.

Set up:



→ Boundary Conditions (B.C.)

$$\begin{array}{llll} (i) & V=0 & when & y=0 \\ (ii) & V=0 & when & y=a \\ (iii) & V=V_0 & when\ x=0 & 0<y<a/2 \\ & V=-V_0 & when\ x=0 & a/2<y<a \\ (iv) & V\to 0 & as & x\to\infty \end{array}$$

Execute:

Clearly, no z-dependence $\Rightarrow V=V(x,y)=X(x)Y(y)$

$$\frac{\partial^2}{\partial x^2}(X(x)Y(y))+\frac{\partial^2}{\partial y^2}(X(x)Y(y))=0$$

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0 \quad\Rightarrow\quad \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{C_1=k^2}+\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{C_2=-k^2}=0$$

C_1 is positive to match B.C. (iv)

$$\begin{array}{lll} X(x) &=& Ae^{kx}+Be^{-kx} \qquad B.C.\ iv:\ V(x)\to 0,\ x\to\infty \therefore A=0 \\ Y(y) &=& C\sin ky+D\cos ky \qquad B.C.\ i:\ V(y)=0,\ y=0 \therefore D=0 \end{array}$$

$$\Rightarrow \boxed{V(x,y)}=Be^{-kx}C\sin ky=\boxed{Ce^{-kx}\sin ky},\quad \boxed{C\equiv B\cdot C}$$

B.C. (ii): $V=0$ when $y=a \therefore \sin ka=0$

$$ka=n\pi \quad\therefore\quad \boxed{k=\frac{n\pi}{a}} \quad (n=1,2,3,\ldots)$$

So, we actually have an infinite number of possible solutions, but we still have to match boundary conditions (iii)

$$\Rightarrow\quad V(x,y)=\sum_{n=1}^{\infty}C_n e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)$$

For $x=0$ $\quad V(0,y)=\underbrace{\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi y}{a}\right)=V_0(y)}_{Fourier\ series}$

C_n are Fourier coefficients.

Dirichlet's Theorem: Any function can be expressed in such series.

To get C_n multiply both sides of the above equation by $\sin(n'\pi y/a)$ and integrate $\quad (1\le n'\le n)$

$$\Rightarrow\int_0^a\sum C_n\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy=\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy$$

$$=\sum C_n\underbrace{\int_0^a\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy}_{solution\ below}=\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy$$

$$\int_0^a\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy=\begin{cases}0 & if\quad n\neq n' \\ \frac{a}{2} & if\quad n=n'\end{cases}$$

Proof on the next page

From Table of integrals:

$$\int\sin^2(bx)dx=\frac{x}{2}-\frac{\sin(2bx)}{4b}+C$$ To be used when n=n'

$$\int\sin(bx)\sin(cx)dx=\frac{\sin[(b-c)x]}{2(b-c)}-\frac{\sin[(b+c)x]}{2(b+c)}+C \Leftarrow b^2\neq c^2$$ To be used when n is not =n'

$n\neq n'$, $b=\frac{n'\pi}{a}$, $c=\frac{n\pi}{a} \Rightarrow \int_0^a\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)dy=$

$$=\left|\frac{\sin\left[\frac{(n'-n)\pi y}{a}\right]}{\frac{2(n'-n)\pi}{a}}-\frac{\sin\left[\frac{(n'+n)\pi y}{a}\right]}{\frac{2(n'+n)\pi}{a}}\right|_0^a=\boxed{\frac{\sin[(n'-n)\pi]}{\frac{2(n'-n)\pi}{a}}-\frac{\sin[(n'+n)\pi]}{\frac{2(n'-n)\pi}{a}}=0.}$$

$n=n'$, $b=\frac{n'\pi}{a} \Rightarrow \int_0^a\sin^2\left(\frac{n'\pi y}{a}\right)dy=\left.\frac{y}{2}-\frac{\sin\left(\frac{2n'\pi y}{a}\right)}{4\frac{n'\pi}{a}}\right|_0^a=$

$$\boxed{=\frac{a}{2}-\frac{\sin(2n'\pi)}{4\frac{n'\pi}{a}}=\frac{a}{2}}.$$

→ All that's left is the term with $n=n'$

$$\Rightarrow\quad C_{n'}\left(\frac{a}{2}\right)=\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy \qquad \boxed{\Rightarrow\quad C_{n'}=\frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy}$$

So, apply B.C. (iii):

$$C_n=\frac{2}{a}\left[\int_0^{a/2}dyV_0\sin\left(\frac{n\pi y}{a}\right)+\int_{a/2}^{a}dy(-V_0)\sin\left(\frac{n\pi y}{a}\right)\right]$$

$$=\frac{2V_0}{a}\left[\left.\frac{-\cos(n\pi y/a)}{n\pi/a}\right|_0^{a/2}+\left.\frac{\cos(n\pi y/a)}{n\pi/a}\right|_{a/2}^{a}\right]$$

$$=\frac{2V_0}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right)+\cos(0)+\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right]$$

$$=\frac{2V_0}{n\pi}\left[1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\left[1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right]\to\begin{Bmatrix} n=1 & \Rightarrow & 0 \\ n=2 & \Rightarrow & 4 \\ n=3 & \Rightarrow & 0 \\ n=4 & \Rightarrow & 0 \end{Bmatrix}$$

$\Rightarrow\ [\quad]=0$ for n odd or divisible by 4

$\Rightarrow\ [\quad]=4$ for n even but not divisible by 4

$$C_n=\begin{Bmatrix}\frac{8V_0}{n\pi} & n=2,6,10,14,\ldots \\ 0 & otherwise\end{Bmatrix}$$

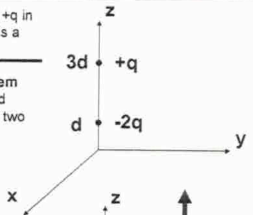
We can write this series as $\boxed{n=4j+2}$ where $j=0,1,2,3,\ldots$

$$\boxed{\Rightarrow\quad V(x,y)=\frac{8V_0}{\pi}\sum_{j=0}^{\infty}e^{-(4j+2)\pi x/a}\frac{\sin\left[(4j+2)\frac{\pi y}{a}\right]}{4j+2}}$$

That's as far as we can go.

PROBLEM 3.6

Find the force on charge +q in (Fig. 3.14) The xy plane is a grounded conductor.



IDENTIFY Relevant concepts

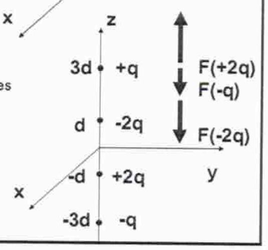
This is a **classic image problem** (See Section 3.2.1) – grounded *xy* plane can be substituted by two mirror charges, +2*q* and –*q*.

Superposition
Coulomb law

SET UP

Draw the diagram including the mirror charges instead of the grounded *xy* plane.

Show the forces on +*q* due to each of the charges.



PROBLEM 3.6 (cont.)

EXECUTE

$$\vec{F} = \vec{F}_{(-2q)} + \vec{F}_{(+2q)} + \vec{F}_{(-q)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right]\hat{z}$$

$$\vec{F} = \frac{q^2}{4\pi\varepsilon_0 d^2}\left(-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right)\hat{z} = -\frac{1}{4\pi\varepsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}.$$

EVALUATE

Force is directed downward ...

EXAMPLE 3.2 & PROBLEM 3.7

Example 3.2: A point charge *q* is situated a distance *a* from the center of the **grounded** conducting sphere of radius *R*.

Find the potential outside the sphere.

Figure 3.12

Problem 3.7: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathfrak{R}_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathfrak{R}_1}+\frac{q_2}{\mathfrak{R}_2}\right)$ (Eq. 3.17)

(a) Using the law of cosines, show that the above equation can be written as:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]$$

where *r* and *θ* are the usual spherical polar coordinates, with the *z* axis along the line through *q*. In this form it is obvious that *V = 0* on the sphere, *r = R*.

(b) Find the induced surface charge on the sphere, as a function of *θ*. Integrate this to get the total induced charge (What it **should** be?)

(c) Calculate the energy of this configuration.

EXAMPLE 3.2 & PROBLEM 3.7 (a), combined

IDENTIFY Relevant concepts

Apply **method of images.**
Find a mirror charge that satisfies same boundary conditions: *V* = 0 at the surface of the sphere, and *V*(∞) = 0.

Apply Eq. (3.17)

A

SET UP Apply the same **trick** as in section **3.2.1.**

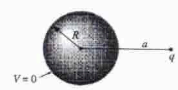


Figure 3.12

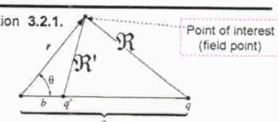


Figure 3.13

Examine a configuration consisting of the point charge *q* together with another point charge $q' = -\frac{R}{a}q$

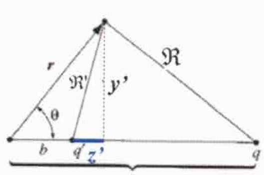
placed a distance $b = \frac{R^2}{a}$ to the right of the center of the sphere.

EXAMPLE 3.2 & PROBLEM 3.7, combined (cont.)

EXECUTE

From Fig. 3.13:

$$z' = r\cos\theta - b$$
$$y' = r\sin\theta$$
$$z'^2 + y'^2 = \mathfrak{R}'^2$$



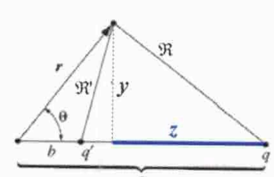
$$\Rightarrow \mathfrak{R}'^2 = r^2\cos^2\theta - 2rb\cos\theta + b^2 + r^2\sin^2\theta =$$
$$= r^2\left(\cos^2\theta + \sin^2\theta\right) + b^2 - 2rb\cos\theta$$
$$\Rightarrow \mathfrak{R}' = \sqrt{r^2+b^2-2rb\cos\theta}$$

{law of cosines}

EXAMPLE 3.2 & PROBLEM 3.7, combined (cont.)

EXECUTE (cont.)

$$z = a - r\cos\theta$$
$$y = r\sin\theta$$
$$z^2 + y^2 = \mathfrak{R}^2$$



$$\Rightarrow \mathfrak{R}^2 = r^2\cos^2\theta - 2ra\cos\theta + a^2 + r^2\sin^2\theta =$$
$$= r^2\left(\cos^2\theta + \sin^2\theta\right) + a^2 - 2ra\cos\theta$$
$$\Rightarrow \mathfrak{R} = \sqrt{r^2+a^2-2ra\cos\theta}$$

EXAMPLE 3.2 & PROBLEM 3.7 (a), combined (cont.)

EXECUTE (cont.)

From Eq. 3.17 $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\Re_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\Re} + \frac{q'}{\Re'}\right)$

$$\Re = \sqrt{r^2 + a^2 - 2ra\cos\theta} \qquad q' = -\frac{R}{a}q \qquad b = \frac{R^2}{a}$$

$$\Re' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{\left(-\frac{R}{a}q\right)}{\sqrt{r^2 + \left(\frac{R^2}{a}\right)^2 - r\frac{R^2}{a}\cos\theta}}\right) =$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + \left(\frac{ar}{R}\right)^2 - 2ra\cos\theta}}\right).$$

General solution (to be used in part (b))

EXAMPLE 3.2 & PROBLEM 3.7 (a), combined (cont.)

EXECUTE (cont.)

$$r = R \;\Rightarrow\; V(R) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}}\right\} = 0.$$

$$V(\infty) = 0.$$

Image charge $q' = -qR/a$ is located at $r = b = R^2/a$.

PROBLEM 3.7 (b,c)

(b) Find the induced surface charge and the total induced charge on the sphere.
(c) Calculate the energy of this configuration.

IDENTIFY Relevant concepts

(b) $\sigma = -\varepsilon_0\frac{\partial V}{\partial n}$ (Eq. 2.49) $at \quad r = R, \quad \frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$

(c) $for\ any \quad z, \quad W = \int_{\infty}^{a}\vec{F}(z)\cdot d\vec{l}$ From Eq. 3.18: $\vec{F}(z) = \frac{1}{4\pi\varepsilon_0}\frac{q^2Rz}{\left(z^2 - R^2\right)^2}.$

EXECUTE (b)

$$\sigma(R,\theta) = -\varepsilon_0\frac{\partial V}{\partial n} = -\varepsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=R} =$$

$$\left.-\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\right)\left\{-\frac{1}{2}\left(r^2 + a^2 - 2ra\cos\theta\right)^{-3/2}(2r - 2a\cos\theta) + \frac{1}{2}\left(R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r - 2a\cos\theta\right)\right\}\right|_{r=R}$$

$$= \frac{q}{4\pi}\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left(R - a\cos\theta - \frac{a^2}{R} + a\cos\theta\right).$$

$$\sigma(\theta) = \frac{q}{4\pi R}\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left(R^2 - a^2\right).$$

PROBLEM 3.7 (b,c) (cont.)

EXECUTE (cont.) (b) Total induced charge on the surface of the sphere:

$$q_{induced} = \int_A \sigma(\theta)dA = \int_0^{2\pi}\int_0^{\pi}\sigma(\theta)\underbrace{R^2 d\phi\sin\theta d\theta}_{dA}$$

$$= \frac{q}{4\pi R}\left(R^2 - a^2\right)R^2\left[\int_0^{2\pi}d\phi\right]\left[\int_{-1}^{1}\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}d(\cos\theta)\right] =$$

$$= \frac{q}{4\pi R}\left(R^2 - a^2\right)R^2 2\pi\left[\frac{2}{2Ra}\left(R^2 + a^2 - 2Rat\right)^{-1/2}\right]_{-1}^{1}, \quad \text{where } \cos\theta = t.$$

$$q_{induced} = \frac{q}{2a}\left(a^2 - R^2\right)\left[\frac{1}{\sqrt{R^2 + a^2 + 2aR}} - \frac{1}{\sqrt{R^2 + a^2 - 2aR}}\right].$$

PROBLEM 3.7 (b,c) (cont.)

(b) continued

EXECUTE (cont.) $a > R \;\Rightarrow\; \sqrt{R^2 + a^2 - 2Ra} = a - R.$

$$q_{induced} = \frac{q}{2a}\left(a^2 - R^2\right)\left[\frac{1}{a + R} - \frac{1}{a - R}\right] = \frac{q}{2a}\left(a^2 - R^2\right)\frac{a - R - a - R}{a^2 - R^2}$$

$$q_{induced} = \frac{q}{2a}(-2R) = -\frac{R}{a}q = q'.$$

(c) Energy is equal to the work done by the force on charge q to bring it from infinity to a. According to the method of images, this force should be equal to the interaction force between q and the induced image charge q'.

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2} \quad \text{(See Figure 3.13)}$$

$$F = \frac{1}{4\pi\varepsilon_0}\left(-\frac{R}{a}q^2\right)\frac{a^2}{\left(a^2 - R^2\right)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{\left(a^2 - R^2\right)^2}$$

$$W = \frac{q^2R}{4\pi\varepsilon_0}\int_{\infty}^{a}\frac{z}{\left(z^2 - R^2\right)^2}dz = \frac{q^2R}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{1}{z^2 - R^2}\right]_{\infty}^{a} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2\left(a^2 - R^2\right)}$$

3.4. Multipole Expansion

Monopole

We know that the potential of a point charge is $V(r)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r}$.

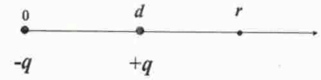
But, what if we had two charges, say of opposite sign. We might expect the potential to approach zero at large r faster than for a point charge.

In fact, for large r, $V(r)\propto\dfrac{1}{r^2}$

This configuration is known as a "dipole".

Strictly a pure dipole would have to have $d\to 0$.
This is sometimes called a "physical dipole".

Dipole



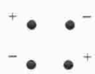
$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{-q}{r}+\frac{q}{r-d}\right)=\frac{1}{4\pi\varepsilon_0}\frac{q(-r+d+r)}{r(r-d)}$$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\frac{qd}{r(r-d)}$$

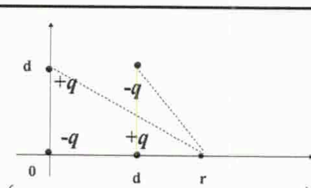
$$r>>d \quad\Rightarrow\quad r-d\approx r \quad\Rightarrow\quad \boxed{V(r)\approx\frac{qd}{4\pi\varepsilon_0}\frac{1}{r^2}}$$

Similarly, there are higher "multipole terms"
For example:

Quadrupole



At large r, the quadrupole potential goes as $V(r)\propto\dfrac{1}{r^3}$



$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{r}+\frac{q}{r-d}+\frac{q}{\sqrt{d^2+r^2}}-\frac{q}{\sqrt{d^2+(r-d)^2}}\right)$$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{qd}{r(r-d)}+\frac{q\left(\sqrt{d^2+(r-d)^2}-\sqrt{d^2+r^2}\right)}{\sqrt{(d^2+r^2)(d^2+(r-d)^2)}}\frac{\left(\sqrt{d^2+(r-d)^2}+\sqrt{d^2+r^2}\right)}{\left(\sqrt{d^2+(r-d)^2}+\sqrt{d^2+r^2}\right)}\right)$$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{qd}{r(r-d)}+\frac{q\left(d^2+(r-d)^2-d^2-r^2\right)}{\sqrt{(d^2+r^2)(d^2+(r-d)^2)}}\frac{1}{\left(\sqrt{d^2+(r-d)^2}+\sqrt{d^2+r^2}\right)}\right)$$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{qd}{r(r-d)}+\frac{q\left(d^2+(r^2-2rd+d^2)-d^2-r^2\right)}{\left(\sqrt{[d^2+(r-d)^2][d^2+r^2]}\right)\left(\sqrt{d^2+(r-d)^2}+\sqrt{d^2+r^2}\right)}\right)$$

$r>>d$

$$V(r)\approx\frac{qd}{4\pi\varepsilon_0}\left(\frac{1}{r(r-d)}+\frac{d-2r}{2r^3}\right)$$

$$\boxed{V(r)\approx\frac{qd}{4\pi\varepsilon_0}\frac{3rd}{2r^4}\propto\frac{1}{r^3}}$$

Quadrupole

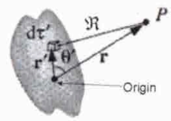
Given any charge distribution, $\rho(r)$, we can always **EXPAND** the potential into a sum of multipole terms.

i.e.

$$V(r)=\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r^3}+\ldots$$

Consider an arbitrary charge distribution, $\rho(r')$.

$$V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{1}{\Re}\rho(r')d\tau'$$



We can expand the $\frac{1}{\Re}$ term in a **power series**:

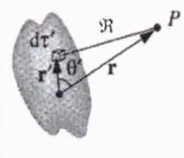
$$\frac{1}{\Re}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$ (Proof on the next two pages.)

where P_n is a Legendre polynomial.

Expansion of $\frac{1}{\Re}$

$$\Re^2=r^2+r'^2-2rr'\cos\theta'=$$

$$=r^2\left[1+\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\cos\theta'\right]$$



$$\Rightarrow\quad \Re=r\sqrt{1+\varepsilon}$$

$$\varepsilon\equiv\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)$$

$$r>>r'\quad\Rightarrow\quad\frac{1}{\Re}=\frac{1}{r}(1+\varepsilon)^{-1/2}=\frac{1}{r}\left(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2-\frac{5}{16}\varepsilon^3+\ldots\right)$$

(binomial expansion)

$$\frac{1}{\Re}=\frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r}-2\cos\theta'\right)^2-\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r}-2\cos\theta'\right)^3+\ldots\right]$$

$$\frac{1}{\Re}=\frac{1}{r}\left[1+\left(\frac{r'}{r}\right)\cos\theta'-\frac{1}{2}\left(\frac{r'}{r}\right)^2+\frac{3}{2}\left(\frac{r'}{r}\right)^2\cos^2\theta'-\frac{3}{2}\left(\frac{r'}{r}\right)^3\cos\theta'+\frac{3}{8}\left(\frac{r'}{r}\right)^4\right]+$$

$$+\frac{1}{r}\left[\frac{5}{2}\left(\frac{r'}{r}\right)^3\cos^3\theta'+\ldots\right]$$

Collect together like powers of r'/r

$$\frac{1}{\Re}=\frac{1}{r}\left[1+\left(\frac{r'}{r}\right)(\cos\theta')+\left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta'-1}{2}\right)+\left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta'-3\cos\theta'}{2}\right)+\ldots\right]$$

Identify Legendre Polynomials:

$$P_0(\cos\theta')=1$$
$$P_1(\cos\theta')=\cos\theta'$$
$$P_2(\cos\theta')=\frac{3\cos^2\theta'-1}{2}$$
$$P_3(\cos\theta')=\frac{5\cos^3\theta'-3\cos\theta'}{2}$$

$$\Rightarrow\quad\frac{1}{\Re}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

So we can then write the potential, $V(r)$ as an expansion in $\frac{1}{r}$

(Eq. 3.95)

$$V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{1}{\Re}\rho(r')d\tau'=\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int(r')^n P_n(\cos\theta')\rho(r')d\tau'$$

Including expanded Legandre's polynomials:

(Eq. 3.96)

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{1}{r}\int\rho(r')d\tau'}_{\text{monopole term}}+\underbrace{\frac{1}{r^2}\int r'\cos\theta'\rho(r')d\tau'}_{\text{dipole term}}+\underbrace{\frac{1}{r^3}\int(r')^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho(r')d\tau'}_{\text{quadrupole term}}+\ldots\right]$$

This series is known as a **multipole expansion**.

3.4.2. Monopole and Dipole Terms

The first term is the **monopole term**.

$$V_{mon}(r)=\frac{1}{4\pi\varepsilon_0}\frac{1}{r}\underbrace{\int\rho(r')d\tau'}_{Q}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

So, if the total charge (net charge) is zero,
the monopole term is also zero.

The next term is the **dipole term**

$$V_{dip}(r)=\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(r')d\tau'$$

$$r'\cos\theta'=\vec{r}'\cdot\hat{r}$$

$$\Rightarrow\quad V_{dip}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}\cdot\int\vec{r}'\rho(\vec{r}')d\tau$$

If we define a **dipole moment** $\vec{p}\equiv\int\vec{r}'\rho(r')d\tau'$

then we can write

$$V_{dip}(r)=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

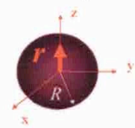
Note: multipole moments are expansion in a power series of $\frac{1}{r}$
→ location of the origin may effect the value of the multipole term.

Problem 3.26.

A sphere of radius R centered at the origin has

$$\rho(r,\theta)=\frac{\kappa R}{r^2}(R-2r)\sin\theta$$

Set up:



Find approximate $V(z)$.

Identify: Eq. (3.96) and $V(z)=V(r\to z)$

Execute:

(*i*) **Monopole term**

$$Q=\int\rho d\tau=\kappa R\int\left(\frac{1}{r^2}(R-2r)\sin\theta\right)\underbrace{\left(r^2\sin\theta drd\theta d\phi\right)}_{d\tau}$$

$$=\kappa R\int(R-2r)\sin^2\theta drd\theta d\phi$$

Note $\int_0^R(R-2r)dr=Rr-r^2\Big|_0^R=0$

$$\Rightarrow\quad Q=0\quad\rightarrow\quad \boxed{V_{mon}=0}$$

(*ii*) **Dipole term**

$$\int r\cos\theta\rho(r)d\tau=\kappa R\int r\cos\theta\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta drd\theta d\phi$$

Note θ integral

$$\int_0^\pi\sin^2\theta\cos\theta d\theta=\frac{\sin^3\theta}{3}\Big|_0^\pi=\frac{1}{3}(0-0)=0\quad\Rightarrow\quad\boxed{V_{dip}=0}$$

(*iii*) **Quadrupole term** $\int r'^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho(r')d\tau'$

After some calculus ... (see next two pages)

$$\int\ =\frac{\kappa\pi^2R^5}{48}\qquad\Rightarrow\qquad\boxed{V_{quad}=\frac{1}{4\pi\varepsilon_0}\frac{\kappa\pi^2R^5}{48z^3}}$$

Quadrupole term: $4\pi\varepsilon_0V_{quad}(r)=\frac{1}{r^3}\int r^2\left(\frac{3}{2}\cos^2-\frac{1}{2}\right)\rho(r,\theta)d\tau.$

$$\int r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho(r,\theta)d\tau=\kappa R\iiint_{r,\theta,\phi}\left\{r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\left[\frac{1}{r^2}(R-2r)\sin\theta\right]\right\}r^2\sin\theta drd\theta d\phi=$$

$$=\boxed{\frac{1}{2}}\kappa R\left[\int_0^R r^2(R-2r)dr\right]\left[\int_0^\pi(3\cos^2\theta-1)\sin^2\theta d\theta\right]\left[\int_0^{2\pi}d\phi\right]$$

$$\int_0^R r^2(R-2r)dr=R\frac{r^3}{3}\Big|_0^R-2\frac{r^4}{4}\Big|_0^R=\frac{R^4}{3}-\frac{R^4}{2}=\boxed{-\frac{R^4}{6}}.$$

$$3\cos^2\theta-1=3(1-\sin^2\theta)-1=2-3\sin^2\theta.$$

$$\int_0^\pi(3\cos^2\theta-1)\sin^2\theta d\theta=\int_0^\pi(2\sin^2\theta-3\sin^4\theta)d\theta.$$

From Tables of Integrals:

$$\int\sin^2\theta d\theta=\frac{\theta}{2}-\frac{\sin2\theta}{4},\quad and\quad\int\sin^4\theta d\theta=\frac{3\theta}{8}-\frac{3\sin2\theta}{16}-\frac{\sin^3\theta\cos\theta}{4}.$$

$$\int(3\cos^2\theta-1)\sin^2\theta d\theta=2\left(\frac{\theta}{2}-\frac{\sin2\theta}{4}\right)\Big|_0^\pi-3\left(\frac{3\theta}{8}-\frac{3\sin2\theta}{16}-\frac{\sin^3\theta\cos\theta}{4}\right)\Big|_0^\pi=2\frac{\pi}{2}-3\left(\frac{3\pi}{8}\right)\boxed{=-\frac{\pi}{8}}.$$

$$\int_0^{2\pi}d\phi=\boxed{2\pi}$$

$$\int r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho(r,\theta)d\tau=\frac{1}{2}\kappa R\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)2\pi=\frac{\kappa\pi^2R^5}{48}.$$

$$\boxed{V_{quad}(z)=\frac{1}{4\pi\varepsilon_0}\frac{\kappa\pi^2R^5}{48z^3}}$$

$$\boxed{V(z)=V_{mono}(z)+V_{dipole}(z)+V_{quad}(z)+\ldots}$$

$$\boxed{=0+0+\frac{1}{4\pi\varepsilon_0}\frac{\kappa\pi^2R^5}{48z^3}+\ldots\cong\frac{1}{4\pi\varepsilon_0}\frac{\kappa\pi^2R^5}{48z^3}}$$