(Chapter 4) **Electric Fields in Matter**

4.1. Polarization

4.1.1. Dielectrics

When an electric field is applied to matter with bound charges (dielectrics), the charge distribution is altered slightly (compared to a dramatic change for conductors).

4.1.2. Induced Dipoles

For example, if an electric field $\vec{E}$ is applied to an atom, the center of + charge, and the center of - charge will be shifted very slightly.

The electric field is said to have induced a dipole moment.

$$\vec{p} = \alpha \vec{E}$$

For weak electric field

Dipole Moment — **Atomic Polarizability** (characteristic for every atom)

Depending on the structure of the charge distribution, polarization is easier in one particular direction. This is almost always the case with molecules.

Therefore, in general,

$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$
$$p_y = \alpha_{xy}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$
$$p_z = \alpha_{xz}E_x + \alpha_{yz}E_y + \alpha_{zz}E_z$$

α_{ij} - polarizability **tensor**

Sometimes it is just a matter of the directions // and $\perp$ to the applied field.

$$\vec{p} = \alpha_{\perp}\vec{E}_{\perp} + \alpha_{//}\vec{E}_{//}$$

4.1.3. Alignment of Polar Molecules

Some charge distributions already have a permanent dipole moment (as opposed to an induced one).

As a result, even though $\sum F = 0$ (for a neutral charge distribution), there is still a **torque**, $\vec{N}$.

For a dipole $\vec{p} = q\vec{d}$ in a (uniform) electric field: $\vec{N} = \vec{p} \times \vec{E}$

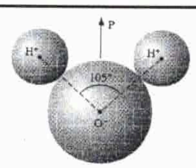


Figure 4.4

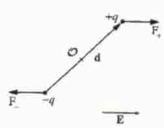


Figure 4.5

Uniform $\vec{E}$ $\quad \vec{F}_{+q} = q\vec{E} \quad \vec{F}_{-q} = -q\vec{E} \quad \sum \vec{F} = 0$

However, torque is

$$\vec{N} = (\vec{r}_{+q} \times \vec{F}_{+q}) + (\vec{r}_{-q} \times \vec{F}_{-q}) = \left[\left(\frac{\vec{d}}{2}\right) \times (q\vec{E})\right] + \left[\left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})\right] = q\vec{d} \times \vec{E}$$

$$\vec{N} = \vec{p} \times \vec{E}$$

Non-uniform $\vec{E}$ $\quad \vec{E}_{at+q} \neq \vec{E}_{at-q}$

$$\vec{F} = \vec{F}_{+q} + \vec{F}_{-q} = q\left(\vec{E}_{at+q} - \vec{E}_{at-q}\right) = q\left(\Delta\vec{E}\right)$$

$$\sum F \neq 0$$

$$\vec{F} = \left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$$

At a point with position vector $\vec{r}$:

$$\vec{N} = \left(\vec{p} \times \vec{E}\right) + \left(\vec{r} \times \vec{F}\right)$$

Problem 4.5 Consider Fig. 4.6

$\vec{p}_1$ and $\vec{p}_2$ are (perfect) dipoles a distance r apart.

Find:
(a) Torque on $\vec{p}_1$ due to $\vec{p}_2$?
(b) Torque on $\vec{p}_2$ due to $\vec{p}_1$?

(Calculate torque on dipole about its own center.)

p_1 Fig. 4.6 p_2

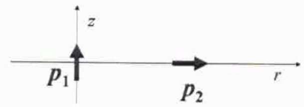
IDENTIFY relevant concepts $\vec{N} = \vec{p} \times \vec{E}_{dip}$ $(Eq. 4.4)$

Eq. (3.99) leads to coordinate free form of electric field:

$$\vec{E}_{dip} = -\vec{\nabla} \cdot V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}] \qquad (Eq. 3.104)$$

PROBLEM 4.5

SET UP Draw diagram



$\vec{p}_1 = p_1\hat{k}$
$\vec{p}_2 = p_2\hat{r}$

EXECUTE

$$(a) \quad \vec{E}_{due\ to\ dip2} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(\vec{p}_2\cdot\hat{r})\hat{r} - \vec{p}_2] =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(p_2\hat{r}\cdot\hat{r})\hat{r} - p_2\hat{r}] =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[(3p_2 - p_2)\hat{r}] = \frac{1}{4\pi\varepsilon_0}\frac{2p_2}{r^3}\hat{r}.$$

PROBLEM 4.5 (continued)

EXECUTE (continued)

$$(a) \quad \vec{N}_1 = \vec{p}_1 \times \vec{E}_{due\ to\ dip2} = (p_1\hat{k}) \times \left(\frac{2p_2}{4\pi\varepsilon_0 r^3}\right)\hat{r} =$$

$$= \frac{p_1 p_2}{2\pi\varepsilon_0 r^3}(\hat{k}\times\hat{r}) \Rightarrow \textit{directed in the page} \ \otimes$$

$$(b) \quad \vec{E}_{due\ to\ dip1} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(p_1\hat{k}\cdot\hat{r})\hat{r} - p_1\hat{k}\right]$$

$$\vec{E}_{due\ to\ dip1} = \frac{1}{4\pi\varepsilon_0 r^3}\left[(0)\hat{r} - p_1\hat{k}\right] = -\frac{p_1}{4\pi\varepsilon_0 r^3}\hat{k}$$

$$\vec{N}_2 = \vec{p}_2 \times \vec{E}_{due\ to\ dip1} = (p_2\hat{r}) \times \left(-\frac{p_1}{4\pi\varepsilon_0 r^3}\hat{k}\right)$$

$$\vec{N}_2 = \frac{p_1 p_2}{4\pi\varepsilon_0 r^3}(\hat{k}\times\hat{r}) \quad \Rightarrow \textit{directed in the page} \ \otimes$$

4.1.4. Polarization

When an electric field is applied to a dielectric, dipoles are induced and/or permanent dipoles align with the field.

Result: lots of aligned dipoles!

The dielectric is said to be polarized.

We define the **polarization**:

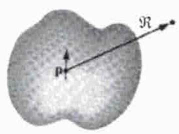
$$\vec{P} = \frac{\textit{dipole moment}}{\textit{unit volume}}$$

4.2. The Field of Polarized Object

4.2.(1&2). Bound Charges and Physical Interpretatrion

Suppose we have a material with $\vec{P} \neq 0.$

It contains many micro-dipoles lined up in the direction of external $\vec{E}$.
All those little dipoles produce an electric field.



For a field point at $\mathfrak{R}$ from a single dipole the potential is:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{\mathfrak{R}}}{\mathfrak{R}^2} \quad (recall\ 3.99)$$

Then total potential is:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{p}(r')\cdot\hat{\mathfrak{R}}}{\mathfrak{R}^2}d\tau'$$

After some vector calculus it becomes (on the next page):

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\oint_S \frac{\sigma_b}{\mathfrak{R}}da' + \int_V \frac{\rho_b}{\mathfrak{R}}d\tau'\right]$$

$$\sigma_b \equiv \vec{P}\cdot\hat{n} \quad (surface\ char.\ density) \quad (4.11)$$
$$\rho_b \equiv -\vec{\nabla}\cdot\vec{P} \quad (volume\ char.\ density) \quad (4.12)$$

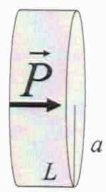
$$Q_{b(surface)} = \int \sigma_b da$$
$$Q_{b(volume)} = \int \rho_b d\tau$$

Now we can treat the problem just like any other one with charges using, for example, Gauss's Law.

These charges $(\sigma_b \cdot A \ \text{and} \ \rho_b \cdot V)$ are referred to as "**bound charges**"
They are **real**! We say they are "bound" because they are not easily removed, but the distribution is analogous to placing σ on the surface of an object and ρ inside the volume of the object.

PROBLEM 4.11.

Problem 4.11 A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization **P**, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$, (ii) for $L \ll a$, and (iii) for $L \approx a$. [This device is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]



PROBLEM 4.11

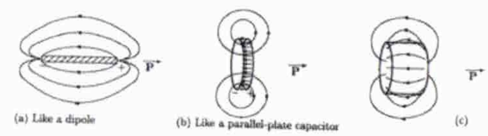
IDENTIFY relevant concepts

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$uniform \equiv const.$$

SET UP (a) L>>a (b) L<<a (c) L~a



(a) Like a dipole (b) Like a parallel-plate capacitor (c)

PROBLEM 4.11 (continued)

EXECUTE

$$P = cont.$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

$$Q_{b(volume)} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \pm P$$

$$Q_{b(surface)} = \int \sigma_b dA = P\pi a^2$$

4.3. Electric Displacement

4.3.1. Gauss's Law in the Presence of Dielectrics

Consider the total charge density within dielectric:

$$\rho_b + \rho_f$$

ρ_b - Bound charge density

ρ_f - Free charge density (free electrons or ions)

Then the differential form of Gauss' Law reads

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_b + \rho_f}{\varepsilon_0} \quad \Rightarrow \quad \varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho_b + \rho_f$$

and, as always $\vec{\nabla}\times\vec{E} = 0$

Since the bound charge results from a polarization, $\rho_b = -\vec{\nabla}\cdot\vec{P}$

We can write: $\varepsilon_0\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{P} + \rho_f$

$$\vec{\nabla}\cdot\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f \qquad (4.22)$$

If we define the electric displacement $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

then Gauss' Law can be written $\vec{\nabla}\cdot\vec{D} = \rho_f$

or in integral form $\oint \vec{D}\cdot d\vec{a} = Q_{f\ enclosed}$

4.3.2. A Deceptive Parallel

Note that these are macroscopic field equations, the microscopic field equations could be quite complex.

Be careful about the displacement. For an electric field we said that because $\vec{\nabla}\times\vec{E} = 0$ we can write $\vec{E} = -\vec{\nabla}V$

(When CURL is zero, then electric field can be presented as GRADIENT of a scalar f-n.)

But … in general $\vec{\nabla}\times\vec{D} \neq 0$ because $\vec{\nabla}\times\vec{P} \neq 0$

$$\vec{D} \neq \vec{\nabla}V'$$

4.3.3. Boundary Conditions

Recall from Chapter 2 that there is a discontinuity of Electric Field (Eq. 2.31) below and above the surface:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0} \qquad (2.31)$$

$$E^{//}_{above} = E^{//}_{below} \qquad (2.32) \qquad E^{//}_{above} - E^{//}_{below} = 0$$

For the electric displacement we have:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \qquad (4.26)$$

$$D^{//}_{above} - D^{//}_{below} = P^{//}_{above} - P^{//}_{below} \neq 0 \qquad (4.27)$$

4.4. Linear Dielectrics

4.4.1. Origin of Polarization

(Susceptibility, Permittivity, Dielectric Constant)

As we noted earlier, Polarization arises when an electric field lines up dipoles. How a material reacts to the application of an electric field depends on the material.

For some dielectrics, termed linear dielectrics the polarization is simply proportional to electric field.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

χ_e is "electric susceptibility" (dimensionless)

$\vec{E}$ is the TOTAL FIELD, not just the external field.

$\Rightarrow$ The nearby dielectric material causes a polarization too.

Note: In some dielectrics, the direction of $\vec{P}$ is not the same as $\vec{E}$.

Here we need to consider the components of a susceptibility tensor

$$\begin{matrix} \chi_{exx} & \chi_{exy} & \cdots \\ \chi_{eyx} & \cdots & \cdots \\ \chi_{ezx} & \cdots & \cdots \end{matrix}$$

Such a medium is anisotropic.

When we write $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

we are considering a linear, isotropic (same in all directions), homogeneous (same in all positions) medium.

So, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$

$$\vec{D} = \varepsilon \vec{E}$$

permittivity

$$\varepsilon \equiv \varepsilon_0 (1 + \chi_e)$$

Permittivity of free space

We can also define a relative permittivity or dielectric constant:

$$K = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$$

If there are no boundaries where χ_e changes, then

$$\vec{E} = \frac{1}{K} \vec{E}_{vac}$$

The field is simply reduced by a factor of K.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Sources are bound charges.

Sources are free and bound charges.

Sources are free charges. (See Eq. 4.22)

PROBLEM 4.32

A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the sphere? Where is the compensating negative bound charge located?

IDENTIFY relevant concepts

Gauss's law in the presence of dielectrics $\oint \vec{D} \cdot d\vec{a} = Q_{f\,enclosed}$ (4.23)

Linear dielectric: $\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}$ (4.32)

and $\vec{P} = \varepsilon_0 \chi_0 \vec{E}$ (4.30)

Bound charges:

$\sigma_b \equiv \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r}$ (4.11) and $\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$ (4.12)

Compensating negative charge (net bound charge): $Q_{comp} = \int \rho_b d\tau$

Also (recall 3-dimensional delta function): $\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(\vec{r})$ (1.99)

PROBLEM 4.32 (cont.)

SET UP & EXECUTE

$$\int \vec{D} \cdot d\vec{a} = Q_{f\,enclosed} \Rightarrow |\vec{D}|(4\pi r^2) = q$$

$$\vec{D} = \frac{q}{4\pi r^2} \hat{r} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{q}{4\pi \varepsilon_0 (1 + \chi_e) r^2} \hat{r}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \chi_e \frac{q}{4\pi \varepsilon_0 (1 + \chi_e)} \frac{\hat{r}}{r^2} = \frac{q}{4\pi} \left(\frac{\chi_e}{1 + \chi_e} \right) \frac{\hat{r}}{r^2}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = \frac{-q}{4\pi} \left(\frac{\chi_e}{1 + \chi_e} \right) \left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right) = -q \left(\frac{\chi_e}{1 + \chi_e} \right) \delta^3(\vec{r})$$

$$\sigma_b = \vec{P} \cdot \hat{r} = \frac{q}{4\pi R^2} \left(\frac{\chi_e}{1 + \chi_e} \right)$$

PROBLEM 4.32 (cont.)

EXECUTE (cont.)

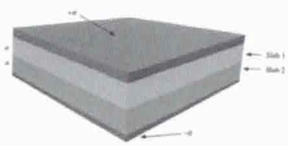
$$Q_{sphere\,surface} = 4\pi R^2 \sigma_b = q \left(\frac{\chi_e}{1 + \chi_e} \right)$$

$$Q_{comp} = \int \rho_b d\tau = -q \left(\frac{\chi_e}{1 + \chi_e} \right) \int \delta^3(\vec{r}) d\tau = -q \left(\frac{\chi_e}{1 + \chi_e} \right)$$

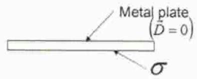
The compensating negative charge is at the center

Problem 4. 18

Consider a parallel plate capacitor filled with two slabs of dielectric material. Each slab has a thickness a, and dielectric constants K_1=2 and K_2=1.5. Free charge density on top (bottom) plate is σ $(-\sigma)$.



(a) Find $\vec{D}$ on each slab.



$$\int \vec{D}\cdot d\vec{a} = Q_{f\ enclosed}$$

$$DA = \sigma A \quad \Rightarrow \quad D = \sigma \ (direction\ down)$$

Similarly for bottom slab:

$$\Rightarrow \quad D = \sigma \quad (direction\ down)$$

(b) Find the electric field in each slab. $\vec{D} = \varepsilon\vec{E}$

$$\Rightarrow \quad \vec{E}_1 = \frac{\sigma}{\varepsilon_1}(down) = \frac{\sigma}{2\varepsilon_0} \quad (top\ slab)$$

$$\vec{E}_2 = \frac{\sigma}{\varepsilon_2}(down) = \frac{2\sigma}{3\varepsilon_0} \quad (bottom\ slab)$$

(c) Find the polarization in each slab. $\vec{P} = \varepsilon_0\chi\vec{E}$

$$P_1 = \varepsilon_0\chi\frac{\sigma}{2\varepsilon_0} = \chi\frac{\sigma}{2} \qquad \text{but} \qquad K = 1+\chi \ \Rightarrow \ \chi = K-1$$

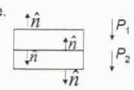
$$P_1 = (2-1)\frac{\sigma}{2} = \frac{\sigma}{2} \qquad P_2 = \varepsilon_0\chi\left(\frac{2\sigma}{3\varepsilon_0}\right) = \frac{2\chi\sigma}{3} = 2(1.5-1)\frac{\sigma}{3} = \frac{\sigma}{3}$$

(d) Find the potential difference between the plates.

$$V = E_1 a + E_2 a \qquad (\vec{E} = \vec{\nabla}\cdot V)$$

$$= \frac{\sigma a}{2\varepsilon_0} + \frac{2\sigma a}{3\varepsilon_0} = \frac{7\sigma a}{6\varepsilon_0}$$

(e) Find the location and amount of all bound charge.



$$(\rho_b = -\vec{\nabla}\cdot\vec{P}) \quad P = cont \ \Rightarrow \ \rho_b = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = -P_1 = -\frac{\sigma}{2} \quad (top\ of\ slab\ 1)$$

$$= +P_1 = +\frac{\sigma}{2} \quad (bottom\ of\ slab\ 1)$$

$$= -P_2 = -\frac{\sigma}{3} \quad (top\ of\ slab\ 2)$$

$$= +P_2 = +\frac{\sigma}{3} \quad (bottom\ of\ slab\ 2)$$

(f) Recalculate the field in each slab and confirm your answer to (b).

In slab 1:

$$\sigma_{above} = \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$$

$$\sigma_{below} = \frac{\sigma}{2} - \frac{\sigma}{3} + \frac{\sigma}{3} - \sigma = -\frac{\sigma}{2} \qquad \Rightarrow \quad E_1 = \frac{\sigma}{2\varepsilon_0}$$

In slab 2:

$$\sigma_{above} = \sigma - \frac{\sigma}{2} + \frac{\sigma}{2} - \frac{\sigma}{3} = +\frac{2\sigma}{3}$$

$$\sigma_{below} = \frac{\sigma}{3} - \sigma = -\frac{2\sigma}{3} \qquad \Rightarrow \quad E_2 = \frac{2\sigma}{3\varepsilon_0}$$

4.4.3. Energy in Dielectric System

Recall from Chapter 2 that the energy of a system of charges can be thought of as being stored in the electric field.

For dielectrics we must modify an original expression for vacuum:

$$W = \frac{1}{2}\varepsilon_0 \int_{all\ space} E^2 d\tau$$

Energy in vacuum

$$\varepsilon_0 \Rightarrow \varepsilon$$

No polarization In vacuum.

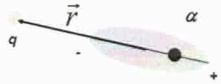
$$\varepsilon\vec{E} = \vec{D}$$

$$\varepsilon E^2 = \vec{D}\cdot\vec{E}$$

$$W = \frac{1}{2}\int_{all\ space} \vec{D}\cdot\vec{E}\,d\tau$$

Energy in dielectric

PROBLEM 4.4 A point charge ***q*** is situated a large distance ***r*** from a neutral atom of polarizability α. Find the force of attraction between them.



PROBLEM 4.4

IDENTIFY Relevant concepts

Induced dipole moment $\vec{p} = \alpha\vec{E}$ (4.1)

Electric field of a point charge $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathfrak{R}^2}\hat{\mathfrak{R}}$ (2.4)

Electric field of a dipole $\vec{E}_{dip}(r,\theta) = \dfrac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$ (3.103)

Force on charge q in an electric field E $\vec{F} = q\vec{E}$ (2.3)

SET UP

$$\hat{\mathfrak{R}} = \hat{r}, \quad \vec{\mathfrak{R}} = \vec{r} \quad \Rightarrow \quad \mathfrak{R}^2 = r^2$$

EXECUTE

$$\vec{E}_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathfrak{R}^2}\hat{\mathfrak{R}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

PROBLEM 4.4 (cont.)

EXECUTE (cont.)

$$\vec{p} = \alpha\vec{E}_q = \frac{\alpha q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\hat{\theta}\right) =$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\frac{\alpha q}{4\pi\varepsilon_0 r^2}\left[2\cos 0(\hat{r}) + \sin 0\,\hat{\theta}\right] =$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{2\alpha q}{r^5}\hat{r} \quad \text{since} \quad \cos 0 = 1, \quad \sin 0 = 0.$$

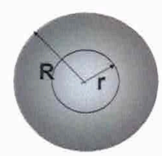
$$\vec{F} = q\vec{E}_{dip} = 2\alpha\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^5}\hat{r}.$$

PROBLEM 4.10 A sphere of radius ***R*** carries a polarization

$$\vec{P}(\vec{r}) = k\vec{r}$$

where ***k*** is a constant and ***r*** is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.
(b) Find the field inside and outside the sphere.



PROBLEM 4.10

IDENTIFY relevant concepts

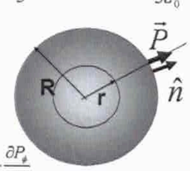
Surface charge density of a polarized object $\sigma_b = \vec{P}\cdot\hat{n}$ (4.11)

Volume charge density of a polarized object $\rho_b = -\vec{\nabla}\cdot\vec{P}$ (4.12)

Electric field inside a uniformly charged sphere (see Prob.2.12) $E(r)4\pi r^2 = \dfrac{1}{\varepsilon_0}\rho\dfrac{4\pi r^3}{3} \Rightarrow \vec{E}(r) = \dfrac{\rho r}{3\varepsilon_0}\hat{r}$

SET UP & EXECUTE

(a) $\sigma_b = \vec{P}(R)\cdot\hat{n} = kR$



$P_r = kr, \quad P_\theta = P_\phi = 0$

$$\vec{\nabla}\cdot\vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial P_\phi}{\partial\phi}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -\frac{1}{r^2}3kr^2 = -3k$$

PROBLEM 4.10 (cont.)

EXECUTE (cont.)

(b) $r < R: \quad \vec{E} = \dfrac{\rho r}{3\varepsilon_0}\hat{r} = \dfrac{(-3k)r}{3\varepsilon_0}\hat{r} = -\dfrac{k}{\varepsilon_0}\vec{r}$

$r > R: \quad \vec{E} = \dfrac{Q_{tot}}{4\pi\varepsilon_0 r^2}\hat{r}$

$$Q_{tot} = \sigma_b\left(4\pi R^2\right) + \rho_b\left(\frac{4}{3}\pi R^3\right)$$

$$Q_{tot} = (kR)\left(4\pi R^2\right) + (-3k)\left(\frac{4}{3}\pi R^3\right) = 4\pi k R^3 - 4\pi k R^3 = 0$$

$$\Rightarrow \vec{E} = 0.$$

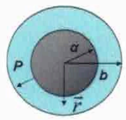
PROBLEM 4.15 A thick spherical shell (inner radius *a*, outer radius *b*) is made of dielectric material with "frozen-in" polarization

$$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r},$$

Where *k* is a constant and *r* is the distance from the center (See Figure below). There is no free charge in the problem. Find the electric field in all three regions by two different methods:

(a) Locate all the bound charge and use Gauss law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find $\vec{D}$, and then get $\vec{E}$ from Eq. 4.21.



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PROBLEM 4.15a

IDENTIFY relevant concepts

Surface charge density of a polarized object $\sigma_b = \vec{P}\cdot\hat{n}$ (4.11)

Volume charge density of a polarized object $\rho_b = -\vec{\nabla}\cdot\vec{P}$ (4.12)

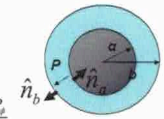
Gauss law

SET UP & EXECUTE

$$P_r = \frac{k}{r}, \quad P_\theta = P_\phi = 0$$

$$\vec{\nabla}\cdot\vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, P_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial P_\phi}{\partial\phi}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$$



$$\hat{n}_a = -\hat{r}, \quad \hat{n}_b = \hat{r} \quad \Rightarrow$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} +\vec{P}\cdot\hat{r} = \dfrac{k}{b} & r = b \\ -\vec{P}\cdot\hat{r} = -\dfrac{k}{a} & r = a \end{cases}$$

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PROBLEM 4.15a (cont.)

EXECUTE (cont.)

$$\vec{E} = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2}\hat{r}$$

$r < a$: $\quad Q_{encl} = 0 \;\Rightarrow\; \vec{E} = 0.$

$a < r < b$: $\quad Q_{encl} = \left(-\frac{k}{a}\right)\left(4\pi a^2\right) + \int_a^r \left(-\frac{k}{r^2}\right)\left(4\pi r^2\right)dr =$

$$= -4\pi ka - 4\pi k(r-a) = -4\pi kr \;\Rightarrow\; \vec{E} = -\frac{k}{\varepsilon_0 r}\hat{r}$$

$r > b$: $\quad Q_{encl} = \left(-\frac{k}{a}\right)\left(4\pi a^2\right) + \left(\frac{k}{b}\right)\left(4\pi b^2\right) + \int_a^b \left(-\frac{k}{r^2}\right)\left(4\pi r^2\right)dr$

$$= -4\pi ka + 4\pi kb - 4\pi k(b-a) = 0 \;\Rightarrow\; \vec{E} = 0$$

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PROBLEM 4.15b

IDENTIFY relevant concepts

Gauss's law in the presence of dielectrics $\oint \vec{D}\cdot d\vec{a} = Q_{f_{encl}}$ (Free enclosed charge)

Electric displacement $\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}$

There are no free charge in the problem.

SET UP & EXECUTE

$$Q_{f_{encl}} = 0 \;\Rightarrow\; \vec{D} = 0$$

$$\Rightarrow\; \varepsilon_0\vec{E} + \vec{P} = 0, \quad \vec{E} = -\frac{1}{\varepsilon_0}\vec{P}$$

$$\Rightarrow\; \vec{E} = \begin{cases} 0 & for\ r < a, r > b \\ -\dfrac{k}{\varepsilon_0 r}\hat{r} & for\ a < r < b \end{cases}$$

EVALUATE

Method (b) is faster, and avoids any explicit reference to bound charges.

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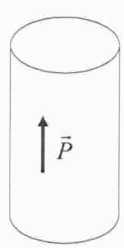
PROBLEM 4.17 For the bar electret of Problem 4.11, make three careful sketches: one of $\vec{P}$, one of $\vec{E}$, and one of $\vec{D}$. Assume *L* is about *2a*. [*Hint:* $\vec{E}$ lines terminate on charges, $\vec{D}$ lines terminate on *free* charges.]



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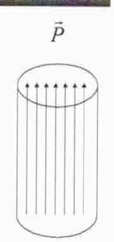
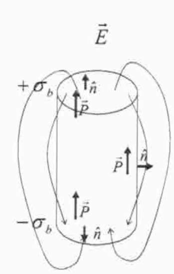
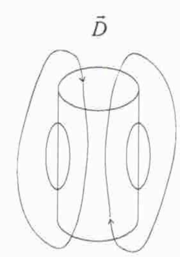
PROBLEM 4.17



Defined by Problem 4.11



There is no volume charge $\left(\vec{\nabla}\cdot\vec{P} = 0\right)$
There are only surface charges on upper and lower circular plates.
On the side
$\sigma_b = \vec{P}\cdot\hat{n} = 0$
Electric field is always directed from positive to negative charge.



Same as $\vec{E}$ *outside*, but lines are not broken since

$$\vec{\nabla}\cdot\vec{D} = 0.$$

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Note on Problem 4.17c

The problem states that there are no free charges anywhere. We can apply the Gauss law for displacement (Eq. 4.22):

$$\vec{\nabla}\cdot\vec{D}=\rho_f \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{D}=0.$$

According to the second potential theorem (see page 54) the displacement vector field in this case (but only in this case!) is a divergence-less or a "solenoidal" field. The equivalent property of this field is that its total flux is zero for any closed surface and that there are no sinks or sources of the field. In this particular case the sources would be the free charges, and they are not present.

Therefore the field lines of the displacement vector field are continuous.

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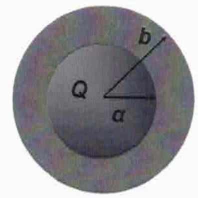
EXAMPLE 4.5 A metal sphere, of radius a, carries a charge Q as shown.

It is surrounded, out to radius b, by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).



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EXAMPLE 4.5

IDENTIFY relevant concepts

Gauss's law in the presence of dielectrics $\oint\vec{D}\cdot d\vec{a}=Q_{f_{encl}}$ (4.23)

Linear dielectric $\vec{D}=\varepsilon\vec{E}=\varepsilon_0(1+\chi_e)\vec{E}$ (4.32)

Potential $V(\vec{r})=-\int_{O}^{r}\vec{E}\cdot d\vec{l}$ (2.21)

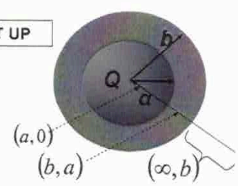
EXECUTE

$\vec{D}\cdot A\hat{r}=Q_{f_{encl}}$

$A=4\pi r^2$

$$\vec{D}=\frac{Q}{4\pi r^2}\hat{r}$$

SET UP



$$\vec{E}=\frac{\vec{D}}{\varepsilon}=\begin{cases}0 & r<a\\ \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} & a<r<b\\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r>b\end{cases}$$

$$V=-\int_{\infty}^{0}\vec{E}\cdot d\vec{l}=-\int_{\infty}^{b}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)dr-\int_{b}^{a}\left(\frac{Q}{4\pi\varepsilon r^2}\right)dr-\int_{a}^{0}(0)dr=\frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b}+\frac{1}{\varepsilon a}-\frac{1}{\varepsilon b}\right)$$

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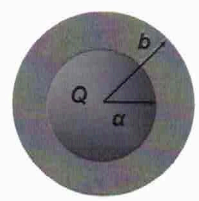
PROBLEM 4.26 A spherical conductor, of radius a, carries a charge Q as shown.

It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.



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PROBLEM 4.26

IDENTIFY relevant concepts

Gauss's law in the presence of dielectrics $\oint\vec{D}\cdot d\vec{a}=Q_{f_{encl}}$ (4.23)

Linear dielectric $\vec{D}=\varepsilon\vec{E}=\varepsilon_0(1+\chi_e)\vec{E}$ (4.32)

Energy in a dielectric system $W=\frac{1}{2}\int\vec{D}\cdot\vec{E}\,d\tau$ (4.58)

SET UP & EXECUTE

From Example 4.5:

$$\vec{D}=\frac{Q}{4\pi r^2}\hat{r} \qquad \vec{E}=\frac{\vec{D}}{\varepsilon}=\begin{cases}0 & r<a\\ \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} & a<r<b\\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r>b\end{cases}$$

$$W=\frac{1}{2}\int\vec{D}\cdot\vec{E}\,\underbrace{d\tau}_{4\pi r^2 dr}=\frac{1}{2}\left(\frac{Q}{4\pi}\right)^2 4\pi\left\{\frac{1}{\varepsilon}\int_a^b\frac{1}{r^2}\frac{1}{r^2}r^2dr+\frac{1}{\varepsilon_0}\int_b^\infty\frac{1}{r^2}\frac{1}{r^2}r^2dr\right\}$$

$$=\frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\}=\frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)$$

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