(Chapter 5) Magnetostatics

5.1 The Lorentz Force Law

5.1.1 Magnetic Fields

Consider a collection of charges.

If charges are at rest, then the problem is one of electrostatics and we need only to consider electric fields.

But, if the charges are in motion, then we have a current and we must also consider magnetic fields.

(Recall "right hand rule" for Magnetic Fields generated by current-carrying conductors – see the figure.)

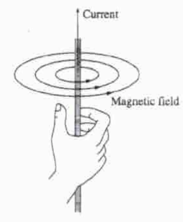


Figure 5.3

5.1.2 Magnetic Forces

Consider a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$:

The magnetic force is given by the Lorentz force law:

$$\vec{F}_{mag} = Q\left(\vec{v}\times\vec{B}\right)$$

And the total force, if there is also an electric field present, is:

$$\vec{F} = Q\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

Note the cross product indicates that:

Magnetic Forces do no work.

$$\vec{F}\perp\vec{v},\vec{B} \quad\Rightarrow\quad W = \int_{\vec{a}}^{\vec{b}}\vec{F}\cdot d\vec{l} = \int_{t_1}^{t_2}\vec{F}\cdot\vec{v}dt = 0$$

5.1.3 Currents

To measure the motion of charge we have to define a current.

A current is the charge per unit time that passes across a surface.

Current is defined such that a positive current is in the direction of motion of the positive charge.

Current (I) has an associated direction.

$\Rightarrow$ $\vec{I}$ is a vector quantity.

Consider a line charge λ, traveling at a velocity $\vec{v}$

$$\vec{I} = \lambda\vec{v}$$

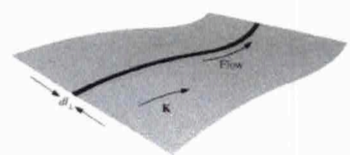
If $d\vec{l}$ is a segment of a wire, then $F_{mag} = \int I\left(d\vec{l}\times\vec{B}\right)$

$$I = const \quad\therefore\quad F_{mag} = I\int d\vec{l}\times\vec{B}$$

λ represents a one dimensional line charge. $\vec{I} = \lambda\vec{v}$

We can also have 2-D and 3-D charge distributions in motion.

We define a surface current density $\vec{K}\equiv\dfrac{d\vec{I}}{dl_\perp} = \sigma\vec{v}$



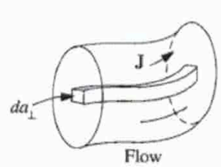
And the Lorentz force law becomes

$$\vec{F}_{mag} = \int\left(\vec{v}\times\vec{B}\right)\sigma da = \int\left(\vec{K}\times\vec{B}\right)da$$

For a volume current, we define "volume current density" as:

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \rho\vec{v}$$

$$\vec{F}_{mag} = \int\vec{v}\times\vec{B}\rho d\tau = \int\vec{J}\times\vec{B}d\tau$$



Note that $\vec{J}$ can be used to calculate

$$I = \int_s\vec{J}\cdot d\vec{a}$$

Then using the divergence theorem: $\int_s\vec{J}\cdot d\vec{a} = \int_v\left(\vec{\nabla}\cdot\vec{J}\right)d\tau$

$$\int_v\left(\vec{\nabla}\cdot\vec{J}\right)d\tau = -\frac{d}{dt}\int_v\rho d\tau \quad\Rightarrow\quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

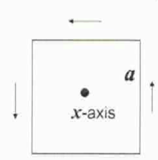
Outward flow decreases the charge left in the volume.

PROBLEM 5.4

Suppose that the magnetic field in some region has the form

$$\vec{B} = kz\hat{x}$$

where k is a constant. Find the force on a square loop (side a), lying on the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.



PROBLEM 5.4

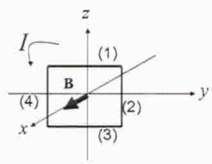
IDENTIFY relevant concepts

Lorentz force law for a current-carrying element $F_{mag} = \int I\left(d\vec{l} \times \vec{B}\right)$

Vector products (Eq. 1.12)

SET UP

Draw a diagram:



Break square loop into four segments, as shown in the diagram.

EXECUTE

(1) $Id\vec{l} = -Idy \cdot \hat{y}; \quad \vec{B} = k\frac{a}{2}\hat{x} \quad \Rightarrow$

$$\vec{F}_{(1)} = I\int d\vec{l} \times \vec{B} = -Ik\frac{a}{2}\int_{-a/2}^{a/2} dy\left(\hat{y} \times \hat{x}\right) = \frac{Ika^2}{2}\hat{z}$$

PROBLEM 5.4 (cont.) EXECUTE (cont.)

(2) $Id\vec{l} = Idz \cdot \hat{z}; \quad \vec{B} = kz\hat{x} \quad \Rightarrow$

$$\vec{F}_{(2)} = I\int d\vec{l} \times \vec{B} = Ik\int_{-a/2}^{a/2} zdz\left(\hat{z} \times \hat{x}\right) = \frac{Ika^2}{2}\hat{y}$$

(3) $Id\vec{l} = Idy \cdot \hat{y}; \quad \vec{B} = -k\frac{a}{2}\hat{x} \quad \Rightarrow$

$$\vec{F}_{(3)} = I\int d\vec{l} \times \vec{B} = -Ik\frac{a}{2}\int_{-a/2}^{a/2} dy\left(\hat{y} \times \hat{x}\right) = \frac{Ika^2}{2}\hat{z}$$

(4) $Id\vec{l} = -Idz \cdot \hat{z}; \quad \vec{B} = kz\hat{x} \quad \Rightarrow$

$$\vec{F}_{(4)} = I\int d\vec{l} \times \vec{B} = -Ik\int_{-a/2}^{a/2} zdz\left(\hat{z} \times \hat{x}\right) = -\frac{Ika^2}{2}\hat{y}$$

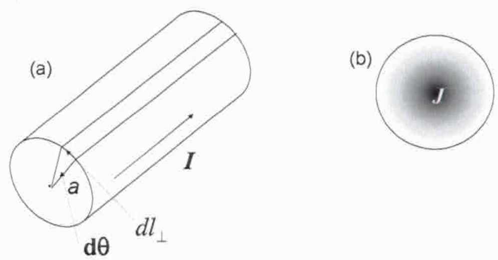
$$\vec{F} = \vec{F}_{(1)} + \vec{F}_{(2)} + \vec{F}_{(3)} + \vec{F}_{(4)} = I\frac{ka^2}{2}\left[(1+1)\hat{z} + (1-1)\hat{y}\right] = Ika^2\hat{z}$$

PROBLEM 5.5

A current ***I*** flows down a wire of radius *a*.

(a) If it is uniformly distributed over the surface, what is the surface current density ***K*** ?

(b) If it is distributed in such a way that the volume current is inversely proportional to the distance from the axis, what is ***J*** ?



PROBLEM 5.5

IDENTIFY relevant concepts

Surface current density $K = \dfrac{dI}{dl_\perp}$

Relation between current and volume current density $I = \int_s \vec{J} \cdot d\vec{a}$

SET UP & EXECUTE

Draw diagram (see previous slide)

(a) $dI = I\dfrac{d\theta}{2\pi}, \quad dl_\perp = ad\theta \qquad K = \dfrac{dI}{dl_\perp} = \dfrac{I}{2\pi a}$

(b) $J \propto \dfrac{1}{r} \quad \Rightarrow \quad let \quad J = \dfrac{\alpha}{r}$

$$I = \int Jda = \int \frac{\alpha}{r} rdrd\theta = \int_0^a \alpha dr \int_0^{2\pi} d\theta = 2\pi\alpha a$$

$$\Rightarrow \quad \alpha = \frac{I}{2\pi a} \quad \Rightarrow \quad J = \frac{I}{2\pi ar}$$

5.2. **The Biot-Savart Law**

Where do magnetic fields come from? $\Longrightarrow$ **Currents**

If the current does not change with time, we say this is a steady current as opposed to a time-varying current.

Analogy: steady charge – electrostatics
steady currents – magnetostatics

The magnetic field produced by a steady current is given by the Biot-Savart Law.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}dl'$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}$$

μ_0 is the permeability of free space $= 4\pi\times10^{-7}\ N/A^2$

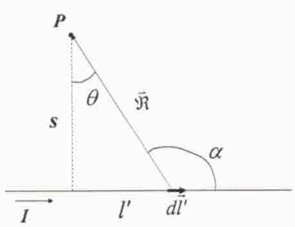
The extension to surface and volume integrals is straightforward:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_s \frac{\vec{K}(\vec{r}\,')\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}da'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}(\vec{r}\,')\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}d\tau' \qquad (5.39)$$

Just like Coulomb's law, superposition applies to magnetic fields too.

EXAMPLE 5.5 Find the magnetic field a distance s from a long straight wire carrying a steady current I.



EXAMPLE 5.5

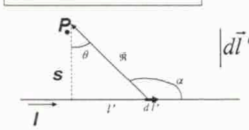
IDENTIFY relevant principles

Biot-Savart law $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}I\int \frac{d\vec{l}'\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}$

Geometrical and trigonometric relations

***SET UP* & EXECUTE**

$$\alpha = \frac{\pi}{2}+\theta$$



$$\left|d\vec{l}'\times\hat{\mathfrak{R}}\right| = dl'\sin\alpha = dl'\cos\theta$$

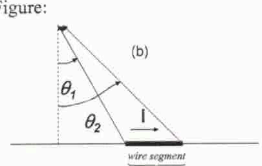
$$l' = s\tan\theta = s\frac{\sin\theta}{\cos\theta} \Rightarrow dl' = s\frac{\cos^2\theta-\left(-\sin^2\theta\right)}{\cos^2\theta}d\theta = \frac{s}{\cos^2\theta}d\theta$$

$$s = \mathfrak{R}\cos\theta \Rightarrow \frac{1}{\mathfrak{R}^2} = \frac{\cos^2\theta}{s^2}$$

EXAMPLE 5.5

SET UP* & EXECUTE *(cont.)

For the wire segment in Figure:



$$\vec{B} = B\hat{k}, \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta d\theta$$

$$B = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)$$

Infinite wire: $\theta_1 = -\frac{\pi}{2}, \quad \theta_2 = \frac{\pi}{2} \Rightarrow$

$$B = \frac{\mu_0 I}{4\pi s}\left[1-(-1)\right] = \frac{\mu_0 I}{2\pi s}$$

EXAMPLE 5.6 Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (see the figure).

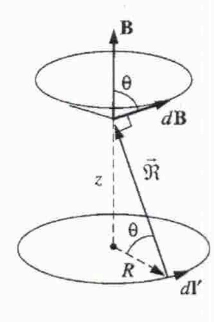


Figure 5.21

EXAMPLE 5.6

SET UP & EXECUTE

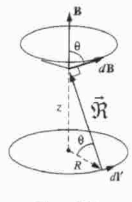


Figure 5.21

$$d\vec{B} = d\vec{B}_{vertical} + d\vec{B}_{horizontal}$$

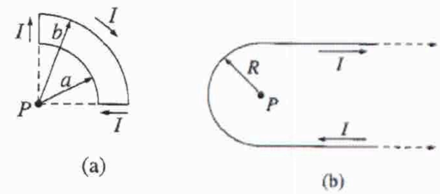
$$\int_{circumference} d\vec{B}_{horizontal} = 0$$

$$B(z) = \int_{circumference} dB_{vertical} = \frac{\mu_0}{4\pi} I \int \frac{dl' \cos\theta}{\Re^2}$$

$$B(z) = \frac{\mu_0}{4\pi}\frac{I\cos\theta}{\Re^2} \int_{circumference} dl' = \frac{\mu_0}{4\pi}\frac{I\cos\theta}{\Re^2} 2\pi R$$

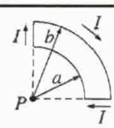
$$\cos\theta = \frac{R}{\Re} = \frac{R}{\sqrt{R^2+z^2}} \quad \Rightarrow \quad B(z) = \frac{\mu_0 I R^2}{2\left(\sqrt{R^2+z^2}\right)^3}$$

PROBLEM 5.9 Find the magnetic field at point *P* for each of steady current configurations shown in the figure below.



PROBLEM 5.9

SET UP & EXECUTE



(a) $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}'\times\hat{r}}{r^2}$

For the sides $d\vec{l} \,//\, \hat{r} \Rightarrow d\vec{l}\times\hat{r} = 0$, sides <u>do not</u> contribute.

From Example 5.6, the magnetic field a distance z above a **circular loop** is $B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$

Here z=0 $\Rightarrow$ $B = \frac{\mu_0 I}{2R}$

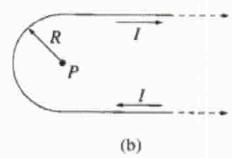
Since we have a quarter segment only, $B = \frac{\mu_0 I}{8R}$. So, $\vec{B} = \vec{B}_a + \vec{B}_b$

Let ⊙ be + $\Rightarrow B = \frac{\mu_0 I}{8a} - \frac{\mu_0 I}{8b}$

$\Rightarrow B = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$ Direction from the page.

PROBLEM 5.9 (cont.) **SET UP & EXECUTE (cont.)**

(b)



The two ½ lines are equivalent to a single infinite line $B = \frac{\mu_0 I}{2\pi R}$

and the ½ circle gives $B = \frac{\mu_0 I}{4R}$.

$$\Rightarrow \quad B(P) = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{4R} = \frac{\mu_0 I}{2R}\left(\frac{1}{\pi} + \frac{1}{2}\right)$$

Direction is into the page, ⊗.

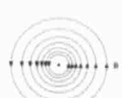
5.3 <u>The Divergence and Curl of Magnetic Field Strength</u>

$$\vec{\nabla}\cdot\vec{B} \quad and \quad \vec{\nabla}\times\vec{B}$$

Consider a wire with a steady current I.

We know that for infinite line $B = \frac{\mu_0 I}{2\pi s}$.

What is the integral $\vec{B}\cdot d\vec{l}$ for fixed s?



$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

This result is actually general, the loop need not be circular and hence if we have multiple wires passing through the loop

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$

This is the integral form of <u>Ampere's Law</u>.

Using Stokes Theorem we know

$$\oint_P \vec{B}\cdot d\vec{l} = \int_s \vec{\nabla}\times\vec{B}\cdot d\vec{a} \qquad (Eq. 1.57)$$

Closed loop — area

Furthermore, $I_{enclosed} = \int_s \vec{J}\cdot d\vec{a}$

$$\Rightarrow \int \vec{\nabla}\times\vec{B}\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$ Differential form of Ampere's Law.

Ampere's Law is always valid. If a system has a certain symmetry it is <u>useful</u> for calculation.

We can now use the Biot-Savart Law for volume current (Eq. 5.39)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathfrak{R}}}{\mathfrak{R}^2} d\tau' \quad \text{to show that} \quad \vec{\nabla}\cdot\vec{B} = 0$$

i.e. the magnetic field is a divergenceless field (no sources or sinks).

$$\vec{\nabla}\cdot\left\{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathfrak{R}}}{\mathfrak{R}^2} d\tau'\right\} \Rightarrow \vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}\right)d\tau'$$

Recall vector identity No. 6 (inside the cover page):

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}\right) = \frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}\right)$$

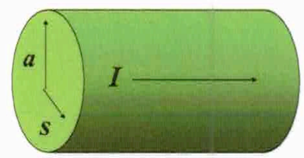
$$\Rightarrow \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}\right) = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0. \qquad \left\{\begin{matrix} \vec{\nabla}\times\vec{J}(\vec{r}\,') = 0 & \vec{J}(\vec{r}\,') \neq f(\vec{r}) \\ \vec{\nabla}\times\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2} = 0 & (see\ \ prob. 1.62) \end{matrix}\right\}$$

PROBLEM 5.13

A steady current I flows down a long cylindrical wire of radius a (see the fig.). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.



PROBLEM 5.13

IDENTIFY relevant concepts	Ampere's law	$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

SET UP & EXECUTE

(a) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

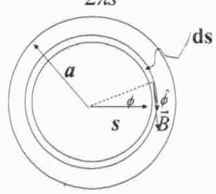
$B(2\pi s) = \mu_0 I_{enclosed}$

$$\Rightarrow \quad \vec{B} = 0 \quad s < a; \qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad s > a$$

(b) $J \propto s \quad \Rightarrow J = ks$

$dA = (2\pi s)ds$

$$I = \int \vec{J}\cdot d\vec{A} = \int_0^a ks\cdot(2\pi s)ds$$



PROBLEM 5.13 (cont.)

EXECUTE (cont.)

$$I = \int_0^a ks\cdot(2\pi s)ds = 2\pi k\int_0^a s^2 ds = 2\pi k\left.\frac{s^3}{3}\right|_0^a = 2\pi k\frac{a^3}{3}$$

$$\Rightarrow \quad k = \frac{3I}{2\pi a^3}, \quad and \quad J = \frac{3I}{2\pi a^3}s$$

$$B(2\pi s) = \mu_0 I_{enclosed}$$

$$I_{enclosed} = \int_0^s JdA = \int_0^s k\bar{s}(2\pi\bar{s})d\bar{s} = 2\pi k\frac{s^3}{3} = 2\pi\frac{3I}{2\pi a^3}\frac{s^3}{3} = I\frac{s^3}{a^3}$$

$$\left.\begin{matrix} s < a, & I_{enclosed} = I\frac{s^3}{a^3} \\ s \geq a, & I_{enclosed} = I \end{matrix}\right\} \Rightarrow \vec{B} = \left\{\begin{matrix} \frac{\mu_0 I s^2}{2\pi a^3}\hat{\phi} & for \quad s < a \\ \frac{\mu_0 I}{2\pi s}\hat{\phi} & for \quad s \geq a \end{matrix}\right\}$$

5.4 Magnetic Vector Potential

5.4.1. The Vector Potential

Recall that **if** $\vec{\nabla}\times\vec{F} = 0 \Leftrightarrow \vec{F} = -\vec{\nabla}V \quad (Eqn. 1.103)$ — Scalar potential

Similarly, **if** $\vec{\nabla}\cdot\vec{F} = 0 \Leftrightarrow \vec{F} = \vec{\nabla}\times\vec{A} \quad (Eqn. 1.104)$ — Vector potential

Since $\vec{B} \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$ we can write magnetic field as

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

$\vec{A}$ is known as the **magnetic vector potential.**

Just as V is defined to within any function whose curl is zero (V is gradient of that function),

we can always add to $\vec{A}$ any function whose divergence is zero.

Hence we can always define $\vec{A}$ so that $\boxed{\vec{\nabla}\cdot\vec{A} = 0.}$

Magnetic vector potential could be calculated: $\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathfrak{R}}d\tau'}$

PROBLEM 5.25

(a) By whatever means you can think of, find the vector potential a distance s from an infinite straight wire carrying a current I (*outside* the wire). Check that

$$\vec{\nabla}\cdot\vec{A} = 0 \quad and \quad \vec{\nabla}\times\vec{A} = \vec{B}.$$

(b) Find the magnetic potential *inside* the wire, if it has radius R and the current is uniformly distributed.

PROBLEM 5.25a

IDENTIFY relevant concepts

Ordinarily, magnetic vector potential is co-directional with current:

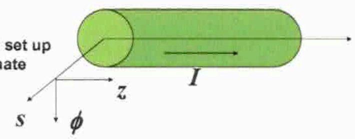
$$\vec{I}(\vec{r}) = I\hat{z} \quad \Rightarrow \quad \vec{A} = A(\vec{r})\hat{z}$$

Magnetic vector potential is defined by:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

SET UP

Draw diagram and set up cylindrical coordinate system ($r \to s$)



EXECUTE

$$\vec{A} = A_z\hat{z} = A(s)\hat{z} \quad \Rightarrow \quad \vec{\nabla}\times\vec{A} = \frac{1}{s}\frac{\partial A(s)}{\partial\phi}\hat{s} - \frac{\partial A(s)}{\partial s}\hat{\phi}$$

PROBLEM 5.25 a(cont.)

EXECUTE (cont.)

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

(Curl in cylindrical coordinates) $(see\ Eq. 1.81)$

$$\Rightarrow \quad \frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s} \quad \Rightarrow \quad A(s) = -\frac{\mu_0 I}{2\pi}\int\frac{ds}{s}$$

$$\Rightarrow \quad \vec{A}(s) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right)\hat{z}$$

a is the integration constant.

EVALUATE

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_z}{\partial z}\hat{z} = \frac{\partial A(s)}{\partial z}\hat{z} = 0$$

(Divergence in cylindrical coordinates) $(see\ Eq. 1.80)$

$$\vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = -\frac{\partial A(s)}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \vec{B}$$

PROBLEM 5.25b

IDENTIFY relevant concepts

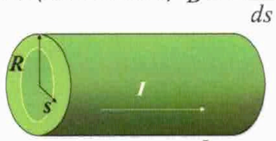
Ampere's law $\oint\vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

Magnetic vector potential has to be continuous at $s = R$.

Current distributed uniformly: $J = const$

Also (see Problem 5.25a): $B = -\dfrac{dA}{ds}\hat{z}$

SET UP & EXECUTE



$$\oint\vec{B}\cdot d\vec{l} = B(2\pi s) = \mu_0 I_{encl} \quad \Rightarrow \quad B = \frac{\mu_0 I_{encl}}{2\pi s}$$

$$I_{encl} = J(\pi s^2) = \frac{I}{\pi R^2}\pi s^2 = I\frac{s^2}{R^2} \quad \Rightarrow \quad \vec{B} = \frac{\mu_0 I}{2\pi}\frac{s}{R^2}\hat{\phi}$$

PROBLEM 5.25b (cont.)

SET UP & EXECUTE (cont.)

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{s}{R^2}\hat{\phi} = -\frac{\partial A}{\partial s}\hat{\phi} \quad \Rightarrow \quad \frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi}\frac{s}{R^2}$$

$$\Rightarrow \quad A(s) = A_z = -\frac{\mu_0 I}{4\pi R^2}s^2 + c$$

$$A_{outside}(R) = A_{inside}(R) \quad \Rightarrow \quad -\frac{\mu_0 I}{2\pi}\ln\left(\frac{R}{a}\right) = -\frac{\mu_0 I R^2}{4\pi R^2} + c$$

a is arbitrary, so we adopt: $\Rightarrow \quad a = R \quad \Rightarrow \quad c = \dfrac{\mu_0 I}{4\pi}.$

$$\Rightarrow \quad \vec{A} = \begin{cases} -\dfrac{\mu_0 I}{2\pi}\ln\left(\dfrac{s}{R}\right)\hat{z} & s \geq R \\ -\dfrac{\mu_0 I}{4\pi R^2}\left[s^2 - R^2\right]\hat{z} & s \leq R \end{cases}$$

5.4.3. Multipole Expansion of the Vector Potential

Just as we expanded $V(r)$ in powers of $1/r$ for a given charge distribution, we can expand the vector potential in powers of $1/r$ for a given current distribution.

Monopole term

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad no\ monopoles$$

$\Rightarrow$ no monopole term in the expansion.

Dipole term

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

m = magnetic dipole moment

$$\vec{m} = I\int d\vec{a}$$