(Chapter 6) **Magnetic Fields in Matter**

6.1 Magnetization

6.1.1. Diamagnets, Paramagnets, Ferromagnets

Just as charges can become "polarized", currents on the microscopic scale can be aligned to cause a "magnetic polarization" or "magnetization".

There are different types of magnetization:

(a) Acquired by external magnetic field

For magnetization parallel to magnetic field
(1) "**paramagnets**"

For magnetization opposite to magnetic field
(2) "**diamagnets**"

(b) Retained in absence of an external magnetic field
(3) "**ferromagnets**"

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6.1.2 Torques and Forces on Magnetic Dipoles

Just like electric dipoles in an electric field, a magnetic dipole $\vec{m}$ in a magnetic field will experience a torque

$(\vec{N} = \vec{p} \times \vec{E})$ $$\vec{N} = \vec{m} \times \vec{B}$$

If magnetic field is **uniform:** $\vec{F} = 0$

For **non-uniform** magnetic field $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$

6.1.4 Magnetization

Just as we defined a net polarization (dipole moment per unit volume) we can define **magnetization:**

$$M = \frac{magnetic\ dipole\ moment}{unit\ volume}$$

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6.2 The Field of Magnetized Object

Given a material with some fixed magnetization $\vec{M}$, if we defined

a bound **volume** current $\vec{J}_b = \vec{\nabla} \times \vec{M}$

and

a bound **surface** current $\vec{K}_b = \vec{M} \times \hat{n}$

we could rewrite the total vector potential:

$$d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{(\vec{M} d\tau) \times \hat{\mathfrak{R}}}{\mathfrak{R}^2} \Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathfrak{R}}}{\mathfrak{R}^2} d\tau'$$

(after several steps of calculus – see p. 264) as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_v \frac{\vec{J}_b(\vec{r}\,')}{\mathfrak{R}} d\tau' + \oint_s \frac{\vec{K}_b(r)}{\mathfrak{R}} da'$$

$\vec{J}_b$ and $\vec{K}_b$ are the bound currents.

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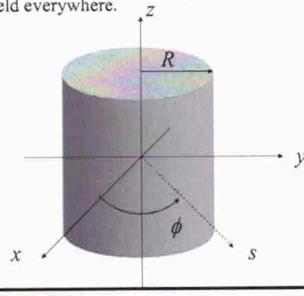
PROBLEM 6.8

A long circular cylinder of radius R carries a magnetization

$$\vec{M} = ks^2 \hat{\phi} \qquad (k = const.)$$

Where s is the distance from the axis and $\hat{\phi}$ is the azimuthal unit vector. Find magnetic field everywhere.



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PROBLEM 6.8

IDENTIFY relevant concepts

Bound volume current $\vec{J}_b = \vec{\nabla} \times \vec{M}$

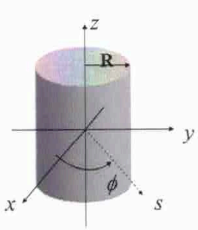
Bound surface current $\vec{K}_b = \vec{M} \times \hat{n}$

Ampere's law $\int \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

SET UP

Cylindrical coordinates

$$\vec{M} = M(s)\hat{\phi}$$ $M_z = M_s = 0$, $\frac{\partial M_\phi}{\partial z} = 0$



EXECUTE

$(\vec{\nabla} \times \vec{M})$ - express in cylindrical coordinates

$$\vec{\nabla} \times \vec{M} = \left(\frac{1}{s}\frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial M_s}{\partial z} - \frac{\partial M_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sM_\phi) - \frac{\partial M_s}{\partial \phi}\right)\hat{z}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sks^2)\hat{z} = 3ks\hat{z}$$

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PROBLEM 6.8 (cont.)

Unit vectors in cylindrical coordinates: $\hat{s} \times \hat{\phi} = \hat{z}$, $\hat{\phi} \times \hat{z} = \hat{s}$, $\hat{z} \times \hat{s} = \hat{\phi}$, $\hat{\phi} \times \hat{s} = -\hat{z}$

EXECUTE (cont.)

$$\vec{K}_b = \vec{M} \times \hat{n} = ks^2 \hat{\phi} \times \hat{s}\Big|_{s=R} = -kR^2 \hat{z}$$

$r < R:$

$$B(2\pi s) = \mu_0 \int_0^s \vec{J}_b \cdot d\vec{a} = \mu_0 \int_0^s (3ks) s\,ds \int_0^{2\pi} d\phi = \mu_0 k 2\pi s^3$$

$$\Rightarrow \quad \vec{B} = \mu_0 ks^2 \hat{\phi} = \mu_0 \vec{M}$$

$r > R:$

$$I_{encl} = \int_0^R \vec{J}_b \cdot d\vec{a} + \int_0^{2\pi R} \vec{K}_b \cdot d\vec{l} = 2\pi kR^3 - kR^2 2\pi R$$

$$I_{encl} = 0 \quad \Rightarrow \quad B = 0$$

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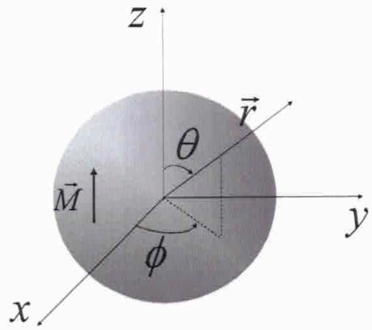
EXAMPLE 6.1

Find the magnetic field of a uniformly magnetized sphere.



EXAMPLE 6.1

Bound currents: $\vec{J}_b = \vec{\nabla} \times \vec{M}$

$\vec{K}_b = \vec{M} \times \hat{n}$

IDENTIFY
Relevant concepts

Analogy with rotating spherical shell (Example 5.11)

SET UP & EXECUTE

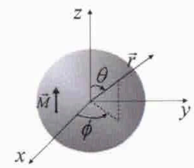
$$M = const \Rightarrow \vec{J}_b = \vec{\nabla} \times \vec{M} = 0$$

$$\left(\begin{array}{l} \textit{Unit vectors in spherical coordinates} \quad \hat{r}, \hat{\theta}, \hat{\phi} \\ \Rightarrow \quad \vec{M} = M\hat{z},\ \hat{n} = \hat{r},\ \hat{z} \times \hat{r} = \sin\theta\hat{\phi} \end{array} \right)$$

$$\vec{K}_b = \vec{M} \times \hat{n} = M \sin\theta\hat{\phi}$$

$$\left(|\vec{K}| = \sigma\omega R \sin\theta \quad \Rightarrow \quad |\vec{B}| = \frac{2}{3}\mu_0 \sigma R \omega \right) \quad (5.68)$$

$$\Rightarrow \quad \vec{B} = \frac{2}{3}\mu_0 \vec{M}$$



6.3 The Auxiliary Field $\vec{H}$

6.3.1 Ampere's law in Magnetized Materials

If we have both "free current" and "bound current", then the total current is $\vec{J} = \vec{J}_f + \vec{J}_b$

Starting from Ampère's law in differential form $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (5.44)

$$\frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + (\vec{\nabla} \times \vec{M})$$

$$\Rightarrow \quad \vec{\nabla} \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M} \right) = \vec{J}_f$$

If we define a quantity $\vec{H}$ so that $\boxed{\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}}$

$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f}$ Differential form of Ampère's law.

$\boxed{\int \vec{H} \cdot d\vec{l} = I_{f\,(encl)}}$ Integral form of Ampère's law.

If $\boxed{\vec{J}_f = 0}$ everywhere, the curl of $\vec{H}$ vanishes (Eq. 6.19), and we can express $\vec{H}$ as:

the gradient of a scalar magnetic potential, W

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f = 0 \quad \Rightarrow \quad \vec{H} = -\vec{\nabla} W.}$$

According to Eq. 6.18 $\left(\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \right)$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot \left(\frac{1}{\mu_0}\vec{B} - \vec{M} \right) = -\vec{\nabla} \cdot \vec{M} \quad (6.23)$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot (-\vec{\nabla} W) = -\vec{\nabla} \cdot \vec{M} \quad \Rightarrow \quad -\nabla^2 W = -\vec{\nabla} \cdot \vec{M}$$

$$\Rightarrow \quad \boxed{\nabla^2 W = \vec{\nabla} \cdot \vec{M}}$$

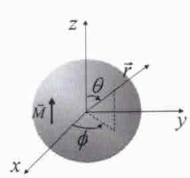
So, W obeys Poisson's Equation $\left(\nabla^2 V = \frac{\rho}{\varepsilon_0} \right)$ with $\vec{\nabla} \cdot \vec{M}$ as the "source".

This opens up all the machinery of Chapter 3.

PROBLEM 6.15

Find the magnetic field inside a uniformly magnetized sphere (Example 6.1) by separation of variables.

Hint: $\vec{\nabla} \cdot \vec{M} = 0$ everywhere except at the surface ($r = R$), so W satisfies Laplace's equation $(\nabla^2 V = 0)$ in the regions $r < R$ and $r > R$.



PROBLEM 6.15

IDENTIFY
relevant
concepts

Solutions of Laplace's equation can be expressed in the terms of Legendre's polynomials (Eqs. 3.78,79)

$$\begin{cases} W_{in}(r,\theta) = \sum A_l r^l P_l(\cos\theta), & (r < R) \\ W_{out}(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r > R) \end{cases}$$

The gradient theorem (Paragraph 2.3.1, page 78)

$$W(\vec{b}) - W(\vec{a}) = \int_a^{\vec{b}} \vec{\nabla} W \cdot d\vec{l}$$

Relations listed on the page 10 – scalar magnetic potential.

PROBLEM 6.15

SET UP — **Boundary conditions:**

$$(i)\quad W(\vec{b})-W(\vec{a})=\int_{\vec a}^{\vec b}\vec{\nabla}W\cdot d\vec{l}=-\int_{\vec a}^{\vec b}\vec{H}\cdot d\vec{l}$$

$$\Rightarrow \lim_{\Delta R\to 0}\left[W_{out}(R+\Delta R,\theta)-W_{in}(R-\Delta R,\theta)\right]=$$

$$=\lim_{\Delta R\to 0}\left[-\int_{R-\Delta R}^{R+\Delta R}\vec{H}\cdot d\vec{l}\right]=0 \quad \boxed{\Rightarrow\ W_{in}(R,\theta)=W_{out}(R,\theta)}$$

Applying this to Eqs. 3.78,79:

$$b.c.(i)\quad A_l R^l=\frac{B_l}{R^{l+1}}\quad\Rightarrow\quad B_l=R^{2l+1}A_l$$

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$$(ii)\quad \vec{\nabla}\cdot\vec{H}=-\vec{\nabla}\cdot\vec{M}\quad\Rightarrow\quad H^{\perp}_{above}-H^{\perp}_{below}=-\left(M^{\perp}_{above}-M^{\perp}_{below}\right)$$

$$\vec{H}=-\vec{\nabla}W,\quad H^{\perp}_{above}=-\frac{\partial W_{out}}{\partial r},\ \ldots.$$

$$\Rightarrow\ \boxed{-\left.\frac{\partial W_{out}}{\partial r}\right|_R+\left.\frac{\partial W_{in}}{\partial r}\right|_R}=-\left(0-M^{\perp}\right)=M^{\perp}=M\hat{z}\cdot\hat{r}=\boxed{M\cos\theta}$$

NO magnetization outside of the sphere.

$$\frac{\partial W_{in}}{\partial r}=\sum_l lA_l r^{l-1}P_l(\cos\theta)$$

$$\frac{\partial W_{out}}{\partial r}=-\sum_l(l+1)\frac{B_l}{r^{l+2}}r^{l-1}P_l(\cos\theta)$$

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PROBLEM 6.15 **EXECUTE**

$$b.c.(ii)\Rightarrow-\left[-\sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta)\right]+\sum lA_lR^{l-1}P_l(\cos\theta)=M\cos\theta$$

$$\Rightarrow\sum\left[(l+1)R^{l-1}A_l+lR^{l-1}A_l\right]P_l(\cos\theta)=M\cos\theta$$

$$\Rightarrow\sum(2l+1)R^{l-1}A_lP_l(\cos\theta)=M\cos\theta$$

Comparing left and right sides: $3A_1=M,\quad A_l=0\ (l\neq 1)$

$$\Rightarrow W_{in}(r,\theta)=\frac{M}{3}r\cos\theta=\frac{M}{3}z$$

$$\vec{H}_{in}=-\vec{\nabla}W_{in}=-\frac{M}{3}\hat{z}=-\frac{1}{3}\vec{M}$$

$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)=\mu_0\left(-\frac{1}{3}\vec{M}+\vec{M}\right)=\boxed{\frac{2}{3}\mu_0\vec{M}}$$

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6.4. Linear and Nonlinear Media

For many materials $\quad\vec{M}\propto\vec{B}$

We define a magnetic **susceptibility** $\boxed{\chi_m}$ so that $\boxed{\vec{M}=\chi_m\vec{H}}$

Which holds for "linear media" since

$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)$$

$$\boxed{=\mu_0\left(1+\chi_m\right)\vec{H}}$$

We can define **permeability**

$$\boxed{\mu\equiv\mu_0\left(1+\chi_m\right)}$$

Then $\quad\boxed{\vec{B}=\mu\vec{H}}$

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PROBLEM 6.18

A sphere of linear magnetic material is placed in an otherwise uniform magnetic field. Find the new magnetic field inside the sphere.

This problem can be solved by at least two different methods:

(a) by method of scalar magnetic potential (See Problem 6.15)

or

(b) by method of successive approximations (See Problem 4.23)

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PROBLEM 6.18 (a) ***By method of scalar magnetic potential***
(See Problem 6.15 and also Example 4.7)

IDENTIFY relevant concepts

$$J_f=0\ \Rightarrow\ \vec{\nabla}\times\vec{H}=0\ \Rightarrow\ \vec{H}=-\vec{\nabla}W\ \Rightarrow\ \nabla^2W=\vec{\nabla}\cdot\vec{M}$$

W **– scalar magnetic potential (expressed in spherical polar coordinates)**

SET UP & EXECUTE

For large r, $\quad\vec{B}(r,\theta)\to\vec{B}_0=B_0\hat{z}\ \Rightarrow\ \vec{H}=\frac{\vec{B}}{\mu_0}=\frac{B_0}{\mu_0}\hat{z}=-\frac{\partial W}{\partial z}\hat{z}$

$$\Rightarrow\ \boxed{W=}\frac{-1}{\mu_0}B_0z=\boxed{-\frac{B_0}{\mu_0}r\cos\theta}$$

Scalar magnetic potential of the outside magnetic field

Using methods of Chapter 3.3 (Eqs. 3.78,79):

$$W_{inside}(r,\theta)=\sum A_lr^lP_l(\cos\theta)\qquad r<R$$

$$W_{outside}(r,\theta)=-\frac{B_0}{\mu_0}r\cos\theta+\sum\frac{B_l}{r^{l+1}}P_l(\cos\theta)\qquad r>R$$

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PROBLEM 6.18 (a) SET UP & EXECUTE (cont.)

Boundary Conditions:

(i) $W_{in}(R,\theta) = W_{out}(R,\theta)$ at the surface of the sphere.

(ii) $$-\mu_0 \frac{\partial W_{out}}{\partial r}\bigg|_R + \mu \frac{\partial W_{in}}{\partial r}\bigg|_R = 0 \qquad \left(B^{\perp}_{above} - B^{\perp}_{below} = 0 \quad Eq.\,6.26\right)$$

$$\Rightarrow \mu_0 \left[\frac{1}{\mu_0} B_0 \cos\theta + \sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta)\right] + \mu \sum l A_l r^{l-1} P_l(\cos\theta) = 0$$

$P_1(\cos\theta) = \cos\theta,\ l = 1$; $\Rightarrow \frac{B_0}{\mu_0}\cos\theta = \frac{B_0}{\mu_0} P_1(\cos\theta)$ } Do not consider this term if $l \neq 1$

For $l \neq 1$

(i) $B_l = R^{2l+1} A_l$

(ii) $[\mu_0(l+1) + \mu l] A_l R^{l-1} = 0 \Rightarrow A_l = 0$

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PROBLEM 6.18 (a) SET UP & EXECUTE (cont.)

For $l = 1$

(i) $\rightarrow$ $A_1 R = -\frac{B_0 R}{\mu_0} + \frac{B_1}{R^2}$

(ii) $\rightarrow$ $B_0 + \frac{2\mu_0 B_1}{R^3} + \mu A_1 = 0$

$\Rightarrow A_1 = \frac{-3B_0}{(2\mu_0 + \mu)}$

$$\Rightarrow W_{in}(r,\theta) = \frac{-3B_0}{(2\mu_0+\mu)} \overbrace{r\cos\theta}^{z}$$

$$\vec{H}_{in} = -\vec{\nabla} W_{in} = -\frac{\partial W_{in}}{\partial z}\hat{z} = \frac{3B_0}{(2\mu_0+\mu)}\hat{z} = \frac{3\vec{B}_0}{(2\mu_0+\mu)}$$

$$\vec{B} = \mu\vec{H} = \frac{3\mu B_0}{(2\mu_0+\mu)} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\vec{B}_0$$

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PROBLEM 6.18 (b) *By method of successive approximations* (See Problem 4.23)

Step 1: B_0 magnetizes the sphere

$$\vec{M}_0 = \chi_m \vec{H}_0 = \chi_m \frac{\vec{B}_0}{\mu_0(1+\chi_m)}$$

This magnetization sets up a field inside the sphere

$$\vec{B}_1 = \frac{2}{3}\mu_0 \vec{M}_0 \qquad \text{(Equation 6.16, Example 6.1)}$$

$$= \frac{2}{3}\mu_0 \frac{\chi_m}{\mu_0(1+\chi_m)} B_0 = \frac{2}{3} k B_0$$

where $k = \frac{\chi_m}{(1+\chi_m)}$

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PROBLEM 6.18 (b)

Step 2: B_1 magnetizes the sphere an additional amount $\vec{M}_1$

$$\vec{M}_1 = \frac{k}{\mu_0}\vec{B}_1$$

but this sets up an additional field in the sphere:

$$\vec{B}_2 = \frac{2}{3}\mu_0 \vec{M}_1 = \frac{2}{3} k \vec{B}_1 = \left(\frac{2}{3}k\right)^2 \vec{B}_0$$

Step 3: B_2 magnetizes the sphere an additional amount $\vec{M}_2$.....

So, the total field is:

$$\Rightarrow \vec{B} = \vec{B}_0 + \vec{B}_1 + \vec{B}_2 + ...$$

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PROBLEM 6.18 (b)

$$\vec{B} = \vec{B}_0 + \frac{2k}{3}\vec{B}_0 + \left(\frac{2k}{3}\right)^2 \vec{B}_0 + ...$$

$$= \left[1 + \left(\frac{2k}{3}\right) + \left(\frac{2k}{3}\right)^2 + ...\right]\vec{B}_0 = \frac{\vec{B}_0}{1 - \frac{2k}{3}}$$

Recall: $k = \frac{\chi_m}{(1+\chi_m)}$

$$\Rightarrow \vec{B} = \frac{1+\chi_m}{1+\chi_m/3}\vec{B}_0$$

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