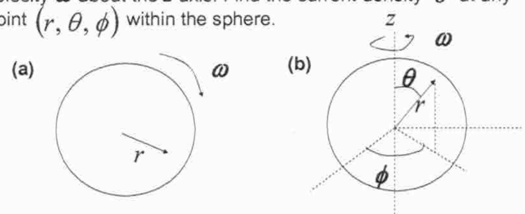
PROBLEM 5.6

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω. What is the surface current density K at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\vec{J}$ at any point (r, θ, ϕ) within the sphere.



PROBLEM 5.6

IDENTIFY relevant concepts

Surface current density $\vec{K} = \sigma \vec{v}$ (5.23)

Volume current density $\vec{J} = \rho \vec{v}$ (5.26)

(σ and ρ are mobile surface and volume charge densities)

Kinematics of rotation

SET UP & EXECUTE

(a) $v = \omega r \Rightarrow K = \sigma v = \sigma \omega r$

(b) $\vec{v} = \vec{\omega} \times \vec{r} = \omega r \sin\theta \hat{\phi}$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$$

$$\vec{J} = \rho \vec{v} = \frac{3Q}{4\pi R^3}\omega r \sin\theta \hat{\phi}$$

PROBLEM 5.10

(a)Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).

Figure 5.24

PROBLEM 5.10a

IDENTIFY relevant concepts

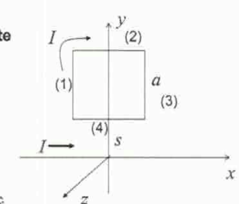
Magnetic field of a long straight wire carrying current I **(in Cartesian coordinates)** $\vec{B} = \frac{\mu_0 I}{2\pi y}\hat{z}$

Force on a segment of *constant* current-carrying wire $\vec{F}_{mag} = I\int\left(d\vec{l} \times \vec{B}\right)$ (5.17)

Force on the square loop $\vec{F}_{tot} = \sum \vec{F}_{mag}$

SET UP

Draw coordinate system



EXECUTE

segment (1): $dl = +dy \Rightarrow$

$$\vec{F}_{mag(1)} = I\int_s^{s+a}\left(dy\hat{y} \times \frac{\mu_0 I}{2\pi y}\hat{z}\right)$$

$$\vec{F}_{mag(1)} = \frac{\mu_0 I^2}{2\pi}\int_s^{s+a}\frac{dy}{y}\hat{x} = \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+a}{s}\right)\hat{x}$$

PROBLEM 5.10a (cont.)

EXECUTE (cont.)

segment (2): $dl = +dx \Rightarrow$

$$\vec{F}_{mag(2)} = I\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(dx\hat{x} \times \frac{\mu_0 I}{2\pi(s+a)}\hat{z}\right)$$

$$\vec{F}_{mag(2)} = -\frac{\mu_0 I^2}{2\pi(s+a)}\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\hat{y} = -\frac{\mu_0 I^2}{2\pi}\frac{a}{s+a}\hat{y}$$

segment (3): $dl = -dy \Rightarrow$

$$\vec{F}_{mag(3)} = -I\int_s^{s+a}\left(dy\hat{y} \times \frac{\mu_0 I}{2\pi y}\hat{z}\right)$$

$$\vec{F}_{mag(3)} = -\frac{\mu_0 I^2}{2\pi}\int_s^{s+a}\frac{dy}{y}\hat{x} = -\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+a}{s}\right)\hat{x}$$

PROBLEM 5.10a (cont.)

EXECUTE (cont.)

segment (4): $dl = -dx \Rightarrow$

$$\vec{F}_{mag(4)} = -I\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(dx\hat{x} \times \frac{\mu_0 I}{2\pi s}\hat{z}\right)$$

$$\vec{F}_{mag(4)} = \frac{\mu_0 I^2}{2\pi s}\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\hat{y} = \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{y}$$

$$\vec{F}_{tot} = \sum \vec{F}_{mag} = \vec{F}_{mag(1)} + \vec{F}_{mag(2)} + \vec{F}_{mag(3)} + \vec{F}_{mag(4)}$$

$$\vec{F}_{tot} = \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+a}{s}\right)\hat{x} - \frac{\mu_0 I^2}{2\pi}\frac{a}{s+a}\hat{y} - \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+a}{s}\right)\hat{x} + \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{y}$$

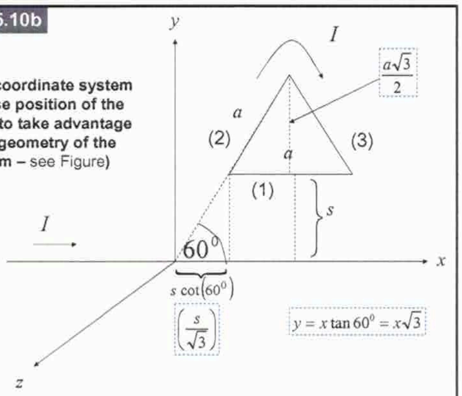
$$\vec{F}_{tot} = -\frac{\mu_0 I^2}{2\pi}\frac{a}{s+a}\hat{y} + \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{y} = \frac{\mu_0 I^2}{2\pi}\frac{-sa+sa+a^2}{s(s+a)}\hat{y}$$

$$\boxed{\vec{F}_{tot} = \frac{\mu_0 I^2 a^2}{2\pi s(s+a)}\hat{y}}$$

PROBLEM 5.10b

SET UP

Draw coordinate system (choose position of the origin to take advantage of the geometry of the problem – see Figure)



PROBLEM 5.10b

EXECUTE

segment (1): $dl = -dx \Rightarrow$

$$\vec{F}_{mag(1)} = -I\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(dx\hat{x}\times\frac{\mu_0 I}{2\pi s}\hat{z}\right)$$

$$\vec{F}_{mag(1)} = \frac{\mu_0 I^2}{2\pi s}\int_{-\frac{a}{2}}^{\frac{a}{2}} dx\hat{y} = \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{y}$$

segment (2): $d\vec{l} = dx\hat{x} + dy\hat{y} \Rightarrow$

$$d\vec{F}_{mag(2)} = I\left(d\vec{l}\times\vec{B}\right) = I\left(dx\hat{x} + dy\hat{y}\right)\times\frac{\mu_0 I}{2\pi y}\hat{z} = \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{y} + dy\hat{x}\right) \Rightarrow$$

$$F_{mag(2)x} = \frac{\mu_0 I^2}{2\pi}\int_s^{s+\frac{a\sqrt{3}}{2}}\frac{dy}{y} = \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+\frac{a\sqrt{3}}{2}}{s}\right) = \frac{\mu_0 I^2}{2\pi}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)$$

PROBLEM 5.10b (cont.)

EXECUTE (cont.)

segment (2, cont.):

$$F_{mag(2)y} = -\frac{\mu_0 I^2}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}}\frac{dx}{y} = -\frac{\mu_0 I^2}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}}\frac{dx}{x\sqrt{3}} = -\frac{\mu_0 I^2}{2\pi\sqrt{3}}\ln\left(\frac{\frac{s}{\sqrt{3}}+\frac{a}{2}}{\frac{s}{\sqrt{3}}}\right)$$

$$F_{mag(2)y} = -\frac{\mu_0 I^2}{2\pi\sqrt{3}}\ln\left(1+\frac{a\sqrt{3}}{2s}\right) \Rightarrow \vec{F}_{mag(2)} = \frac{\mu_0 I^2}{2\pi}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)\left(\hat{x}-\frac{1}{\sqrt{3}}\hat{y}\right)$$

segment (3):

$$d\vec{F}_{mag(3)} = I\left(d\vec{l}\times\vec{B}\right) = I\left(dx\hat{x} - dy\hat{y}\right)\times\frac{\mu_0 I}{2\pi y}\hat{z} = \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{y} - dy\hat{x}\right) \Rightarrow$$

$$F_{mag(3)x} = -\frac{\mu_0 I^2}{2\pi}\int_s^{s+\frac{a\sqrt{3}}{2}}\frac{dy}{y} = -\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{s+\frac{a\sqrt{3}}{2}}{s}\right) = -\frac{\mu_0 I^2}{2\pi}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)$$

PROBLEM 5.10b (cont.)

EXECUTE (cont.)

segment (3, cont.):

$$F_{mag(3)y} = -\frac{\mu_0 I^2}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}}\frac{dx}{y} = -\frac{\mu_0 I^2}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}}\frac{dx}{x\sqrt{3}} = -\frac{\mu_0 I^2}{2\pi\sqrt{3}}\ln\left(\frac{\frac{s}{\sqrt{3}}+\frac{a}{2}}{\frac{s}{\sqrt{3}}}\right)$$

$$F_{mag(3)y} = -\frac{\mu_0 I^2}{2\pi\sqrt{3}}\ln\left(1+\frac{a\sqrt{3}}{2s}\right) \Rightarrow \vec{F}_{mag(3)} = \frac{\mu_0 I^2}{2\pi}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)\left(-\hat{x}-\frac{1}{\sqrt{3}}\hat{y}\right)$$

$$\vec{F}_{tot} = \vec{F}_{mag(1)} + \vec{F}_{mag(2)} + \vec{F}_{mag(3)}$$

$$\vec{F}_{tot} = \frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s}-\frac{2}{\sqrt{3}}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)\right]\hat{y} + \frac{\mu_0 I^2}{2\pi}\left[\ln\left(1+\frac{a\sqrt{3}}{2s}\right)-\ln\left(1+\frac{a\sqrt{3}}{2s}\right)\right]\hat{x}$$

$$\vec{F}_{tot} = \frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s}-\frac{2}{\sqrt{3}}\ln\left(1+\frac{a\sqrt{3}}{2s}\right)\right]\hat{y}$$

PROBLEM 5.11

Find the magnetic field at point P on the axis of a tightly wound solenoid consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (see Figure 5.25).

Express your answer in terms of θ_1 and θ_2. Consider the turns to be essentially circular, and use the result of Example 5.6.

What is the field on the axis of an infinite solenoid (infinite in both directions)?

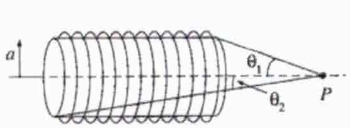


Figure 5.25

PROBLEM 5.11

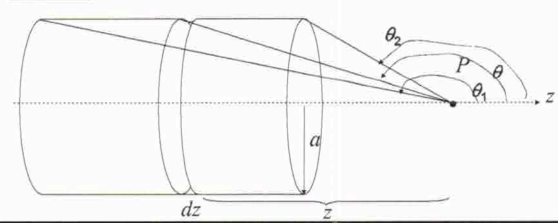
IDENTIFY relevant concepts

Magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (Example 5.6)

$$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}} \quad (5.38)$$

Also: $I \to nIdz$

SET UP



PROBLEM 5.11

EXECUTE

$$B(z)=\frac{\mu_0 nI}{2}\int\frac{a^2}{\left(a^2+z^2\right)^{3/2}}dz$$

$$z=a\cot\theta \Rightarrow dz=-\frac{a}{\sin^2\theta}d\theta$$

$$\sin\theta=\frac{a}{\sqrt{a^2+z^2}} \Rightarrow \frac{1}{\left(a^2+z^2\right)^{3/2}}=\frac{\sin^3\theta}{a^3}$$

$$B=\frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2}a^2\frac{\sin^3\theta}{a^3}\left(-\frac{a}{\sin^2\theta}\right)d\theta=-\frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta d\theta$$

$$B=\frac{\mu_0 nI}{2}\cos\theta\Big|_{\theta_1}^{\theta_2}=\frac{\mu_0 nI}{2}\left(\cos\theta_2-\cos\theta_1\right)$$

$z=\left(-\infty,\infty\right)$

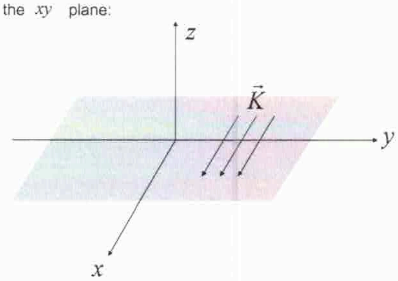
$$\theta_2=0,\quad \theta_1=\pi \Rightarrow B=\frac{\mu_0 nI}{2}\left[1-(-1)\right]=\boxed{\mu_0 nI}$$

EXAMPLE 5.8

Find the magnetic field of an infinite uniform surface current

$$\vec{K}=K\hat{x}$$

flowing over the xy plane:



EXAMPLE 5.8

IDENTIFY relevant concepts

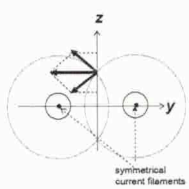
$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r})\times\hat{\mathfrak{R}}}{\mathfrak{R}^2}da' \quad (5.39)$$

$\vec{B}\perp\vec{K} \Rightarrow B_x=0$

$B_z=0 \Rightarrow \vec{B}=B_y\hat{y}$

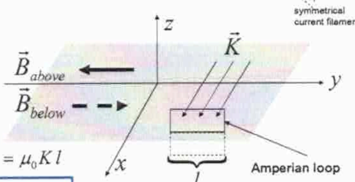
Ampere's law $\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{encl}$

Right hand rule



SET UP

Amperian loop that cuts through the current sheet



EXECUTE

$$\oint\vec{B}\cdot d\vec{l}=2Bl=\mu_0 I_{encl}=\mu_0 Kl$$

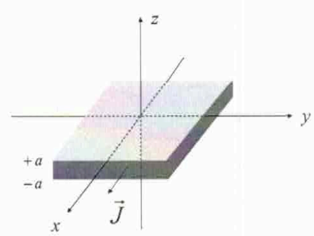
$$\vec{B}=\begin{cases}+\dfrac{\mu_0 K}{2}\hat{y} & z<0\\ -\dfrac{\mu_0 K}{2}\hat{y} & z>0\end{cases} \quad (5.56)$$

PROBLEM 5.14

A thick slab extending from

$$z=-a \quad to \quad z=+a$$

carries a uniform volume current $\vec{J}=J\,\hat{x}$ (see Figure below).
Find the magnetic field $\vec{B}(z)$ both inside and outside the slab.



PROBLEM 5.14

IDENTIFY relevant concepts

Ampere's law

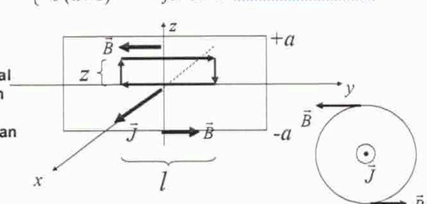
$$\oint\vec{B}\cdot d\vec{l}=\mu_0\int\vec{J}\cdot d\vec{a}=\mu_0 I_{encl}$$ ← The current enclosed by the amperian loop

$$\vec{B}\sim\begin{cases}J\left(\hat{x}\times\hat{z}\right) & for\ z<0\\ -J\left(\hat{x}\times\hat{z}\right) & for\ z>0\end{cases}$$

at $z=0$ $B=0$ (Example 5.8)

SET UP

1. Draw vertical cross-section of the slab
2. Draw amperian loop



EXECUTE

For amperian loop shown in the figure:

$$\oint\vec{B}\cdot d\vec{l}=\underbrace{2Bz\cos\left(\frac{\pi}{2}\right)}_{two\ z-segments}+\underbrace{B(z)l}_{upper\ segment}+\underbrace{B(0)l}_{lower\ segment}=B(z)l$$

PROBLEM 5.14 (cont.)

EXECUTE (cont.)

$$\Rightarrow B(z)l=\mu_0\overbrace{J\underbrace{l\,z}_{area}}^{I_{encl}}$$

$$\Rightarrow \vec{B}=\mu_0 J\,z\left(\hat{x}\times\hat{z}\right)=-\mu_0 J\,z\hat{y} \qquad for \quad -a<z<+a$$

For amperian loop larger than the slab cross-section:

$$\vec{B}=\begin{cases}\mu_0 J\,a\left(\hat{x}\times\hat{z}\right)=-\mu_0 J\,a\hat{y} & for\ z>+a\\ \mu_0 J\,a\left[\hat{x}\times\left(-\hat{z}\right)\right]=\mu_0 J\,a\hat{y} & for\ z<-a\end{cases}$$

PROBLEM 5.16

A large parallel-plate capacitor with uniform surface charge $+\sigma$ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Figure.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?

PROBLEM 5.16

IDENTIFY relevant concepts

Ampere law
Amperian loop
(See Example 5.8)

Current density due to moving surface charge $\vec{K} = \sigma \vec{v}$

Lorentz force law

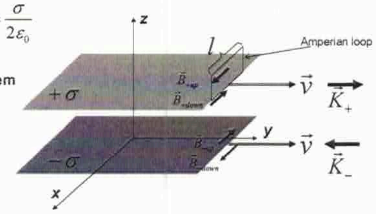
$$\vec{F} = \int \underbrace{\left(\vec{K} \times \vec{B}\right)}_{\text{force per unit area}} da \quad (5.24)$$

Principle of superposition!!

Parallel plate capacitor $E = \dfrac{\sigma}{2\varepsilon_0}$

SET UP

1. Draw coordinate system
2. Draw amperian loop



EXECUTE

(a) From Example 5.8:

$$\vec{B}_{+up} = \frac{\mu_0 K}{2}\hat{x}, \quad \vec{B}_{+down} = -\frac{\mu_0 K}{2}\hat{x} \quad \Rightarrow \quad \vec{B}_{+up} = \frac{\mu_0 \sigma v}{2}\hat{x}, \quad \vec{B}_{+down} = -\frac{\mu_0 \sigma v}{2}\hat{x}$$

$$\vec{B}_{-up} = -\frac{\mu_0 K}{2}\hat{x}, \quad \vec{B}_{-down} = \frac{\mu_0 K}{2}\hat{x} \qquad \vec{B}_{-up} = -\frac{\mu_0 \sigma v}{2}\hat{x}, \quad \vec{B}_{-down} = \frac{\mu_0 \sigma v}{2}\hat{x}$$

PROBLEM 5.16 (cont.)

EXECUTE (cont.)

(a, cont.)

$$\vec{B}_{above} = \vec{B}_{+up} + \vec{B}_{-up} = \frac{\mu_0 \sigma v}{2}(\hat{x} - \hat{x}) = 0$$

$$\vec{B}_{between} = \vec{B}_{+down} + \vec{B}_{-up} = \frac{\mu_0 \sigma v}{2}(-\hat{x} - \hat{x}) = -\mu_0 \sigma v$$

$$\vec{B}_{below} = \vec{B}_{+down} + \vec{B}_{-down} = \frac{\mu_0 \sigma v}{2}(-\hat{x} + \hat{x}) = 0$$

(b) $\vec{f}_{mag} = \vec{K}_{upper} \times \vec{B}_{-up} = \sigma v \hat{y} \times \left(-\dfrac{\mu_0 \sigma v}{2}\hat{x}\right) = \dfrac{\mu_0 \sigma^2 v^2}{2}\hat{z}$

(c) $\vec{f}_e = \sigma \vec{E} = -\dfrac{\sigma^2}{2\varepsilon_0}\hat{z}$

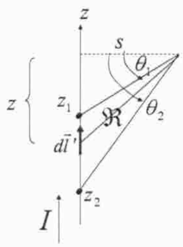
$$\vec{f}_{mag} + \vec{f}_e = 0 \quad \Rightarrow$$

$$\frac{\mu_0 \sigma^2 v^2}{2} = \frac{\sigma^2}{2\varepsilon_0} \quad \therefore \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$$

PROBLEM 5.22 (cont.)

SET UP

1. Draw diagram
2. $d\vec{l}\,' = dz\,\hat{z}$
3. $\mathfrak{R} = \sqrt{z^2 + s^2}$



EXECUTE

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathfrak{R}} d\vec{l}\,' = \frac{\mu_0 I}{4\pi}\int \frac{dz\,\hat{z}}{\sqrt{z^2+s^2}}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz\,\hat{z}}{\sqrt{z^2+s^2}} = \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]_{z_1}^{z_2} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{z}$$

PROBLEM 5.22 (cont.)

EVALUATE

$$A_s = A_\phi = 0,\quad A_z = A(s),\quad \frac{\partial A_z}{\partial \phi} = 0$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A}{\partial s}\hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-\left[(z_2)^2+s^2\right]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{(z_1)^2-\left[(z_1)^2+s^2\right]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\phi}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\phi} = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\phi}$$

PROBLEM 5.26

Find the vector potential above and below the plane surface current in Example 5.8.

IDENTIFY relevant concepts

From Example 5.8:

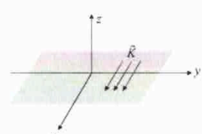
$$\vec{K} = K\,\hat{x} \quad\Rightarrow\quad \vec{B} = \begin{cases} +\dfrac{\mu_0 K}{2}\hat{y} & z<0 \\ -\dfrac{\mu_0 K}{2}\hat{y} & z>0 \end{cases}$$

Magnetic vector potential:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathfrak{R}}da' \quad (5.64) \quad\Rightarrow\quad \vec{A}\parallel\vec{K} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\mathfrak{R} = \mathfrak{R}(z) \quad\Rightarrow\quad \vec{A} = A(z)\hat{x}$$

SET UP



PROBLEM 5.26

EXECUTE

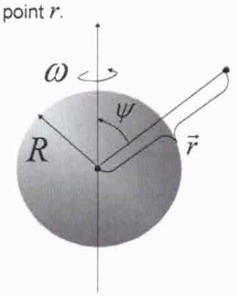
$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A(z) & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\hat{y}$$

$$\vec{B} = \begin{Bmatrix} +\dfrac{\mu_0 K}{2}\hat{y} & z<0 \\ -\dfrac{\mu_0 K}{2}\hat{y} & z>0 \end{Bmatrix} \quad\Rightarrow\quad A(z) = \begin{Bmatrix} +\dfrac{\mu_0 K}{2}z + C & z<0 \text{ or } z=-|z| \\ -\dfrac{\mu_0 K}{2}z + C & z>0 \text{ or } z=|z| \end{Bmatrix}$$

$$\Rightarrow\quad \vec{A} = \left(-\frac{\mu_0 K}{2}|z| + C\right)\hat{x}$$

EXAMPLE 5.11

A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produces at point r.



EXAMPLE 5.11

Magnetic vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')}{\mathfrak{R}}da' \quad (5.64)$$

IDENTIFY relevant concepts

$$\vec{K} = \sigma\vec{v}$$

$$\vec{v} = \vec{\omega}\times\vec{r}\,'$$

SET UP

Set up coordinate system so that:

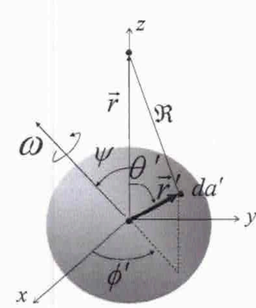
$\vec{r} = r\,\hat{z}$

$\vec{\omega} = \omega\sin\psi\,\hat{x} + \omega\cos\psi\,\hat{z}$

$r' = R$ (spherical shell)

$\mathfrak{R} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$

$da' = R^2\sin\theta' d\theta' d\phi'$



EXAMPLE 5.11 (cont.)

EXECUTE

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{K(\vec{r}')}{\Re}da' \qquad \vec{K} = \sigma\vec{v}$$

$$\vec{v} = \vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$\vec{v} = R\omega\left[(-\cos\psi\sin\theta'\sin\phi')\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{y} + (\sin\psi\sin\theta'\sin\phi')\hat{z}\right]$$

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi}\sigma R^3\omega\cos\psi\left(\int_0^{2\pi}\sin\phi' d\phi'\right)\left(\int_0^{\pi}\frac{\sin^2\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right)\hat{x}$$

$$+\frac{\mu_0}{4\pi}\sigma R^3\omega\left[\begin{matrix}\cos\psi\left(\int_0^{2\pi}\cos\phi' d\phi'\right)\left(\int_0^{\pi}\frac{\sin^2\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right) \\ -\sin\psi\left(\int_0^{2\pi}d\phi'\right)\left(\int_0^{\pi}\frac{\sin\theta'\cos\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right)\end{matrix}\right]\hat{y}$$

$$+\frac{\mu_0}{4\pi}\sigma R^3\omega\sin\psi\left(\int_0^{2\pi}\sin\phi' d\phi'\right)\left(\int_0^{\pi}\frac{\sin^2\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right)\hat{z}$$

EXAMPLE 5.11 (cont.)

EXECUTE (cont.)

$$\int_0^{2\pi}\sin\phi' d\phi' = 0 \Rightarrow A_x = A_z = 0$$

$$\int_0^{2\pi}\cos\phi' d\phi' = 0 \qquad \int_0^{2\pi}d\phi' = 2\pi$$

$$\vec{A}(\vec{r}) = -\overbrace{\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}}^{const.}\left(\int_0^{\pi}\frac{\cos\theta'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right)\hat{y}$$

$$\cos\theta' = u \Rightarrow du = -\sin\theta' d\theta$$

$$\int_0^{\pi}\frac{\cos\theta'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}} = -\int_1^{-1}\frac{udu}{\sqrt{R^2+r^2-2Rru}} = \int_{-1}^{1}\frac{udu}{\sqrt{R^2+r^2-2Rru}}$$

$$= -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\Bigg|_{-1}^{+1}$$

(Look in the table of integrals)

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right] = \Sigma$$

EXAMPLE 5.11 (cont.)

EXECUTE (cont.)

Point r **inside** the sphere:

$$R > r \Rightarrow |R-r| = R-r$$

$$\Sigma_{inside} = -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)(R-r) - (R^2+r^2-Rr)(R+r)\right]$$

$$= -\frac{1}{3R^2r^2}\left[-2rR^2 - 2r^3 + 2R^2r + Rr^2 - Rr^2\right] = \frac{2r^3}{3R^2r^2} = \frac{2r}{3R^2}$$

Point r **outside** the sphere:

$$R < r \Rightarrow |R-r| = r-R$$

$$\Sigma_{outside} = -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)(r-R) - (R^2+r^2-Rr)(r+R)\right]$$

$$= -\frac{1}{3R^2r^2}\left[-2Rr^2 - 2R^3 + 2r^2R + rR^2 - rR^2\right] = \frac{2R^3}{3R^2r^2} = \frac{2R}{3r^2}$$

EXAMPLE 5.11 (cont.)

EXECUTE (cont.)

$$\vec{A}_{inside}(\vec{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}\frac{2r}{3R^2}\hat{y} = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$$

$$\vec{A}_{outside}(\vec{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}\frac{2R}{3r^2}\hat{y} = \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r})$$

Note:

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ 0 & 0 & r \end{vmatrix} = -\omega r\sin\psi\,\hat{y}$$

EXAMPLE 5.11 (cont.)

EXECUTE (cont.)

For spherical coordinates (r, θ, ϕ):

$$\vec{\omega} = \omega\hat{z} = \omega\hat{r} \Rightarrow \vec{r} = r\cos\theta\hat{r} + r\sin\theta\hat{\theta}$$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \omega & 0 & 0 \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix} = \omega r\sin\theta\hat{\phi}$$

$$\vec{A}_{inside}(\vec{r}) = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R\omega\sigma}{3}r\sin\theta\hat{\phi} = A_{\phi i}\hat{\phi}$$

$$\vec{A}_{outside}(\vec{r}) = \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} = A_{\phi o}\hat{\phi}$$

EXAMPLE 5.11 (cont.)

EVALUATE

$$\vec{A}_{inside}(\vec{r}) = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R\omega\sigma}{3}r\sin\theta\hat{\phi} = A_{\phi i}\hat{\phi}$$

$$\vec{A}_{outside}(\vec{r}) = \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} = A_{\phi o}\hat{\phi}$$

$$\vec{B}_{inside} = \vec{\nabla}\times\vec{A}_{inside} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_{\phi i})\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi i})\hat{\theta} \qquad (\hat{\phi}\times\hat{\phi} = 0)$$

$$= \frac{\mu_0 R\omega\sigma}{3}\left[\frac{r(2\sin\theta\cos\theta)}{r\sin\theta}\hat{r} - \frac{2r\sin\theta}{r}\hat{\theta}\right]$$

$$= \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\hat{r} - \sin\theta\hat{\theta})$$

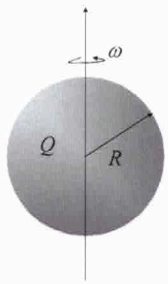
$$\left.\begin{matrix}\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}\end{matrix}\right\} \quad (1.64)$$

$$\Rightarrow \quad \vec{B} = \frac{2}{3}\mu_0\sigma R\omega\hat{z} = \frac{2}{3}\mu_0\sigma R\vec{\omega}$$

Inside the sphere, magnetic field does not depend on position!

PROBLEM 5.29

Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere, of total charge Q and radius R, which is rotating at a constant angular velocity $\boldsymbol{\omega}$.



PROBLEM 5.29

IDENTIFY relevant concepts

Magnetic vector potential inside and outside a uniformly charged spherical shell (Example 5.11, Eq. 5.67)

$$\vec{A}_{inside}(\vec{r}) = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R\omega\sigma}{3} r\sin\theta\hat{\phi} = A_{\phi i}\hat{\phi}$$

$$\vec{A}_{outside}(\vec{r}) = \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} = A_{\phi o}\hat{\phi}$$

Definition of magnetic vector potential $\vec{B} = \vec{\nabla}\times\vec{A}$

Spherical polar coordinates

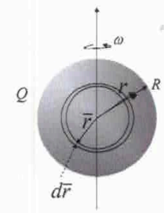
SET UP

1. Decompose the sphere into a series of spherical shells. Then

$$\vec{A} = \int_0^R d\vec{A} = \int_0^r d\vec{A}_{inside} + \int_r^R d\vec{A}_{outside}$$

2. Substitute in Eq. 5.67:

$$R \to \bar{r},\quad \sigma \to \rho d\bar{r},\quad \vec{A} \to d\vec{A}$$



PROBLEM 5.29 (cont.)

EXECUTE

$$\vec{A} = \overbrace{\frac{\mu_0\omega\rho}{3} r\sin\theta\hat{\phi}\int_0^r \bar{r}d\bar{r}}^{inside} + \overbrace{\frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\phi}\int_r^R \bar{r}^4 d\bar{r}}^{outside}$$

$$\vec{A} = \frac{\mu_0\omega\rho}{3}\sin\theta\left[r\frac{r^2}{2} + \frac{1}{r^2}\left(\frac{R^5}{5} - \frac{r^5}{5}\right)\right]\hat{\phi} = \frac{\mu_0\omega\rho}{15r^2}\sin\theta\left(2r^5 + R^5\right)\hat{\phi}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A)\right]\hat{r} \quad - \frac{1}{r}\frac{\partial}{\partial r}(rA)\hat{\theta} \qquad \left(\hat{\phi}\times\hat{\phi} = 0\right)$$

$$\vec{B} = \frac{\mu_0\omega\rho}{15}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{r^2}\sin\theta\left(2r^5 + R^5\right)\right)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r^2}\sin\theta\left(2r^5 + R^5\right)\right)\hat{\theta}\right\}$$

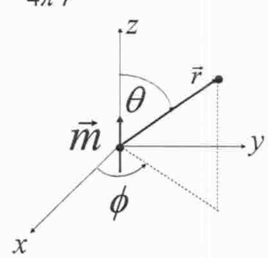
$$\vec{B} = \frac{\mu_0\omega\rho}{15}\left\{2\cos\theta\frac{1}{r^3}\left(2r^5 + R^5\right)\hat{r} - \frac{\sin\theta}{r}\left(8r^3 - \frac{R^5}{r^2}\right)\hat{\theta}\right\}$$

$$\vec{B} = \frac{\mu_0\omega\rho}{15r^3}\left\{2\cos\theta\left(2r^5 + R^5\right)\hat{r} - \sin\theta\left(8r^5 - R^5\right)\hat{\theta}\right\}$$

PROBLEM 5.33

Show that the magnetic field of a dipole can be written in coordinate-free form:

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] \qquad (5.87)$$



PROBLEM 5.33

IDENTIFY relevant concepts

Magnetic field of a dipole at point (r,θ,ϕ)

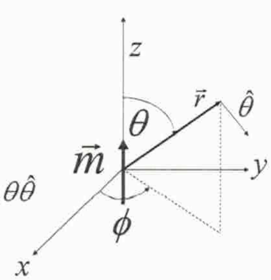
$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \qquad (5.86)$$

SET UP

$$\vec{m}\cdot\hat{r} = m\cos\theta$$

$$\vec{m}\cdot\hat{\theta} = -m\sin\theta$$

$$\vec{m} = m\cos\theta\hat{r} - m\sin\theta\hat{\theta}$$



PROBLEM 5.33

EXECUTE

$\hat{r}\cdot\hat{r} = 1$

$\hat{\theta}\cdot\hat{r} = 0$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\overbrace{\left(m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta}\right)}^{\vec{m}}\cdot\hat{r}\right)\hat{r} - \overbrace{\left(m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta}\right)}^{\vec{m}}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[(3m\cos\theta - 0)\hat{r} - m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m\cos\theta\,\hat{r} - m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

PROBLEM 5.36 Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points

$$r > R$$

the potential is that of a perfect dipole.

IDENTIFY relevant concepts

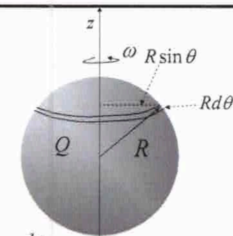
Multipole expansion of the magnetic vector potential:

1. **Monopole term always zero.**
2. **Dipole term:** $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$
3. **Magnetic dipole moment:** $\vec{m} \equiv I\int d\vec{a} = I\vec{a}$ — "vector area" of current loop

If rotation is around z-axis: $\vec{a} = a\hat{z}$, $\vec{m} = m\hat{z}$

PROBLEM 5.36

SET UP



1. **Decompose shell into rings of radius** $R\sin\theta$ **and width** $Rd\theta$

Electric charge of the ring

$$dq = \sigma(2\pi R\sin\theta)Rd\theta$$

Time of one revolution $T = \frac{2\pi}{\omega}$

Current in the ring due to rotation $dI = \frac{dq}{T} = \sigma\omega R^2\sin\theta d\theta$

Magnetic moment of the ring $dm = dI\,a = (\sigma\omega R^2\sin\theta d\theta)[\pi(R\sin\theta)^2]$

EXECUTE $m = \int_{sphere} dm = \sigma\omega R^4\int_0^\pi \sin^3\theta d\theta \Rightarrow \vec{m} = \frac{4\pi}{3}\sigma\omega R^4\hat{z}$

$$\Rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\sigma\omega R^4\frac{\sin\theta}{r^2}\hat{\phi} = \frac{\mu_0\sigma\omega R^4}{3}\frac{\sin\theta}{r^2}\hat{\phi}$$