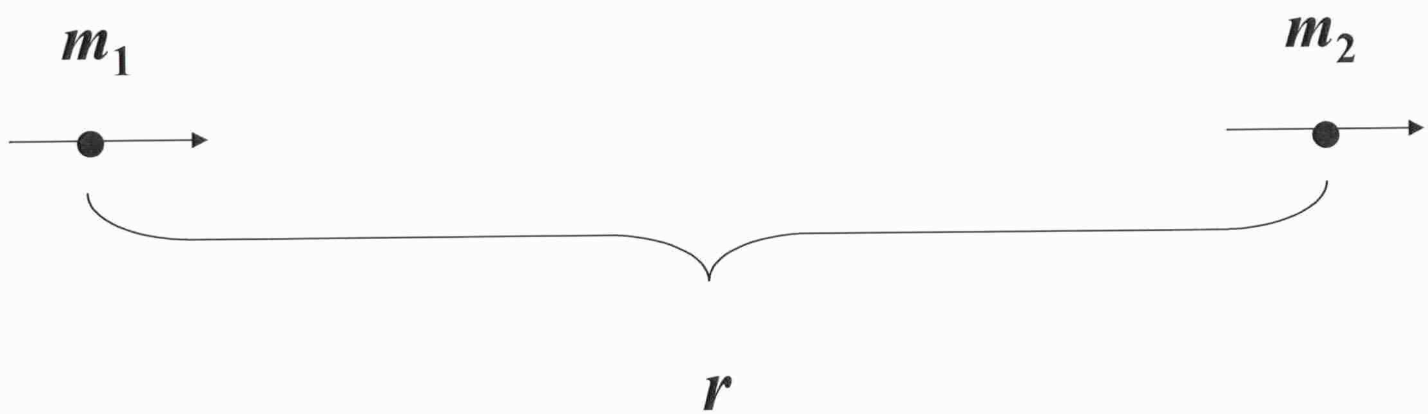
PROBLEM 6.3

Find the force of attraction between two magnetic dipoles, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, oriented as shown in the figure, a distance r apart,

(a) using $F = 2\pi IRB\cos\theta \quad (6.2)$

and

(b) using $\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right) \quad (6.3)$



PROBLEM 6.3

IDENTIFY relevant concepts

Every magnetic dipole is associated with a current loop $(radius\ R, current\ I)$

$$\vec{m} \equiv I \int d\vec{a} = I\vec{a} \qquad (5.84)$$

$$\Rightarrow m = IR^2\pi$$

Magnetic field due to a magnetic dipole is nonuniform, and the magnitude of the force acting on the current loop associated with another dipole:

$$F = 2\pi IRB\cos\theta \qquad (6.2)$$

Magnetic field of a dipole in coordinate-free form (Problem 5.33)

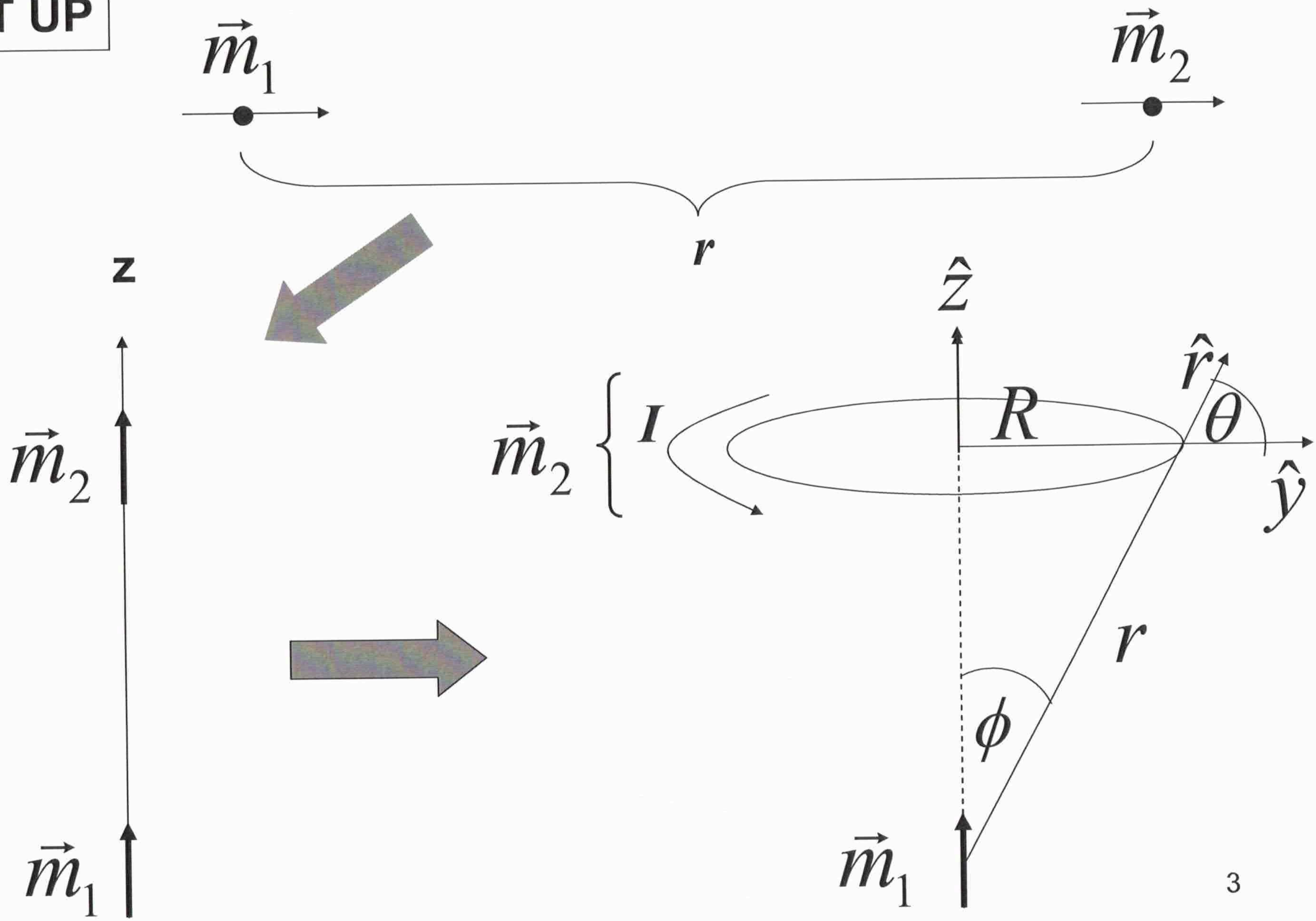
$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] \qquad (5.87)$$

Force acting on a magnetic dipole in a magnetic field

$$\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right) \qquad (6.3)$$

PROBLEM 6.3

SET UP



PROBLEM 6.3a

EXECUTE

$$F = 2\pi IR\left(B\cos\theta\right)$$

Force on the second dipole

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

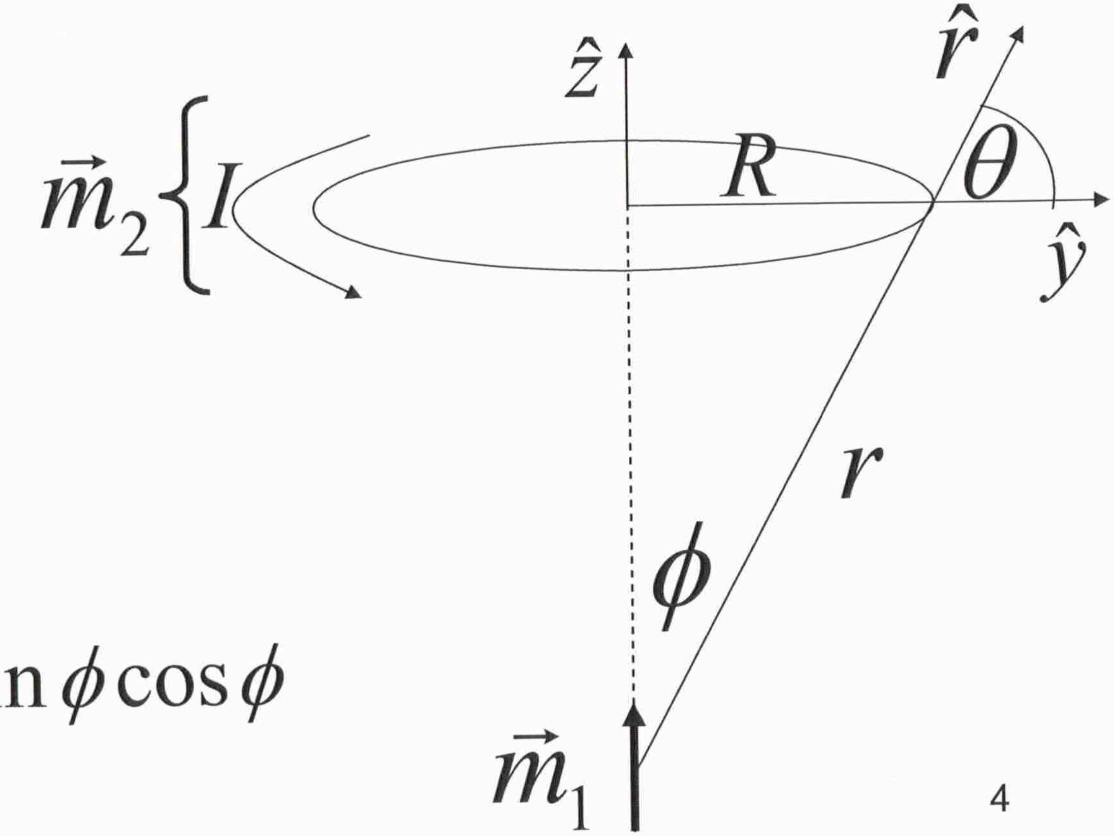
Magnetic field due to first dipole

$$B\cos\theta = \vec{B}\cdot\hat{y} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_1\cdot\hat{r})(\hat{r}\cdot\hat{y}) - (\vec{m}_1\cdot\hat{y})\right]$$

$$\begin{Bmatrix} \vec{m}_1\cdot\hat{y} = 0 \\ \hat{r}\cdot\hat{y} = \sin\phi \\ \vec{m}_1\cdot\hat{r} = m_1\cos\phi \end{Bmatrix}$$

$$\Downarrow$$

$$B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$$



PROBLEM 6.3a

EXECUTE (cont.)

$$B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$$

$$\Rightarrow F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$$

$$\left.\begin{aligned}\sin\phi &= \frac{R}{r}\\ \cos\phi &= \frac{\sqrt{r^2-R^2}}{r}\end{aligned}\right\} \Rightarrow \quad F = 3\frac{\mu_0}{2}m_1IR^2\frac{\sqrt{r^2-R^2}}{r^5}$$

$$\left(m_2 = IR^2\pi \quad \Rightarrow \quad IR^2 = \frac{m_2}{\pi}\right)$$

$$\boxed{F = \frac{3\mu_0}{2\pi}m_1m_2\frac{\sqrt{r^2-R^2}}{r^5}}$$

EVALUATE

$$\boxed{R << r \quad \Rightarrow \quad F = \frac{3\mu_0}{2\pi}\frac{m_1m_2}{r^4}}$$

PROBLEM 6.3b

EXECUTE

$$\vec{F} = \vec{\nabla}\left(\vec{m} \cdot \vec{B}\right) \qquad (6.3)$$

$$\hat{r} \equiv \hat{z}$$

(for 6.3b only)
(no current loop involved as in 6.3a)

$$\Rightarrow \quad \vec{F}_{on\, m2} = \vec{\nabla}\left(\vec{m}_2 \cdot \vec{B}_{m1}\right) = \left(\vec{m}_2 \vec{\nabla}\right) \cdot \vec{B}_{m1}$$

$$\vec{B}_{dip\, m1}\left(\vec{r}\right) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\vec{m}_1 \cdot \hat{r}\right)\hat{r} - \vec{m}_1\right]$$

$$\Rightarrow \vec{F}_{on\, m2} = \left(\vec{m}_2 \vec{\nabla}\right) \cdot \vec{B}_{m1} = \left(m_2 \frac{d}{dz}\right)\left[\frac{\mu_0}{4\pi}\frac{1}{z^3}\underbrace{\left(3\left(\vec{m}_1 \cdot \hat{z}\right)\hat{z} - \vec{m}_1\right)}_{2\vec{m}_1}\right]$$

$$\vec{F} = \frac{\mu_0}{2\pi} m_1 m_2 \hat{z} \frac{d}{dz}\left(\frac{1}{z^3}\right) = -\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{z^4}\hat{z}$$

EVALUATE

since $r = z \quad \Rightarrow \quad \vec{F} = -\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{r^4}\hat{z}$

PROBLEM 6.12

An infinitely long cylinder of radius ***R***, carries a "frozen-in" magnetization, parallel to the axis

$$\vec{M} = ks\hat{z}$$

where ***k*** is a constant and ***s*** is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) As in Sect. 6.2, locate all the bound currents, and calculate the field they produce.

(b) Use Ampere's law (in the form of Eq. 6.20) to find $\vec{H}$ and then get $\vec{B}$ from Eq. 6.18.

PROBLEM 6.12

IDENTIFY relevant concepts

Bound Currents $\begin{cases} volume & \vec{J}_b = \vec{\nabla} \times \vec{M} \\ surface & \vec{K}_b = \vec{M} \times \hat{n} \end{cases}$

Ampere's law for bound currents:

$$\oint_{amperian\ loop} \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} = \mu_0 \left(\int_{loop\ area} \vec{J}_b \cdot d\vec{a} + \oint_{amperian\ loop} \vec{K}_b d\vec{l} \right)$$

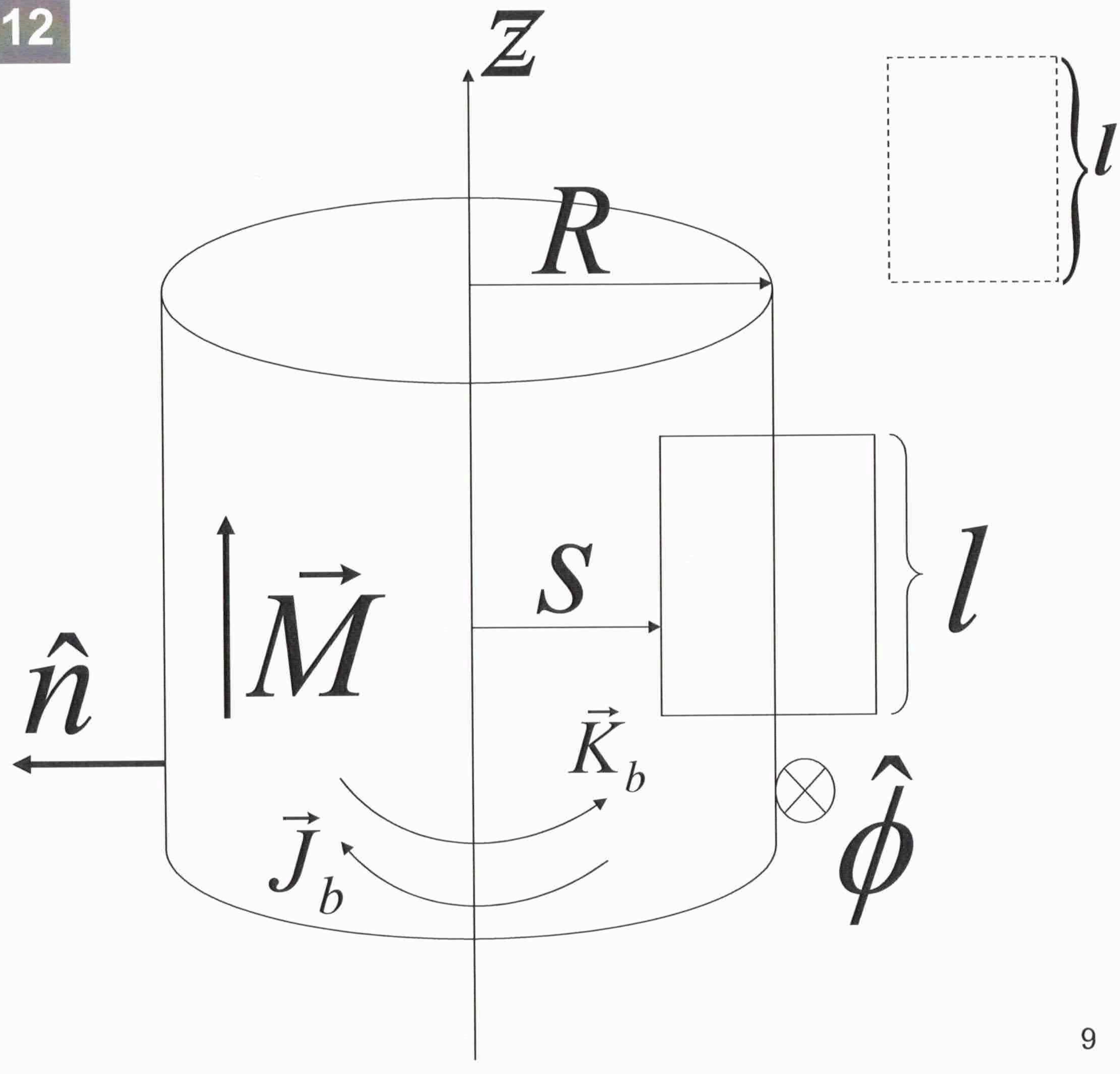
Ampere's law in magnetized materials:

$$\oint_{amperian\ loop} \vec{H} \cdot d\vec{l} = I_{free\ encl} \qquad (6.20)$$

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M} \qquad (6.18)$$

PROBLEM 6.12

SET UP



PROBLEM 6.12a

EXECUTE

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -\frac{\partial M_z}{\partial s}\hat{\phi} = -k\hat{\phi}$$

$$\vec{K}_b = \vec{M} \times \hat{n} = kR\hat{\phi}$$

Amperian loop

Outside:

No currents outside of the cylinder $\Longrightarrow \quad \vec{B}_{outside} = 0$

Inside:

$$\oint \vec{B} \cdot d\vec{l} = B(s)l = \mu_0 I_{encl} = \mu_0 \left[\int_{amp.loop} J_b da + K_b l \right]$$

$$= \mu_0 \left[-kl(R-s) + kRl \right] = \mu_0 kls$$

$$\Longrightarrow \quad \vec{B}_{inside} = \mu_0 ks\, \hat{z}$$

PROBLEM 6.12b

EXECUTE

No free currents $I_{free\,encl} = 0$

Amperian loop: $\oint \vec{H} d\vec{l} = Hl = \mu_0 I_{free\,encl} = 0$

$$\Rightarrow \quad \vec{H} = 0 \quad \Rightarrow \quad \vec{B} = \mu_0 \vec{M}$$

Outside: $\vec{M} = 0 \quad \Rightarrow \quad \vec{B}_{outside} = 0$

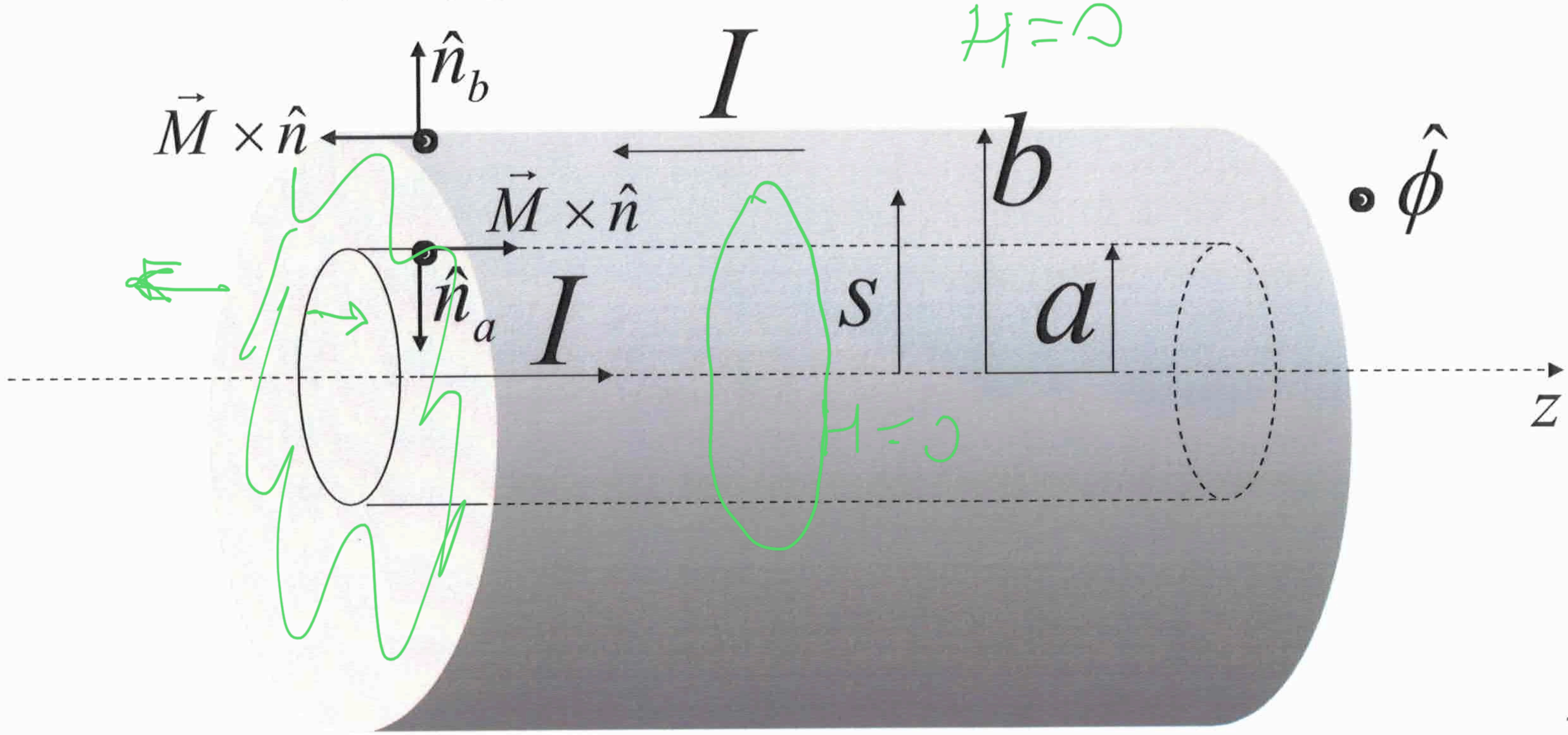
Inside:

$$\vec{M} = ks\,\hat{z} \quad \Rightarrow \quad \vec{B}_{inside} = \mu_0 ks\,\hat{z}$$

PROBLEM 6.16

A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m.

A current I flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface. Find the magnetic field in the region between the tubes. As a check, calculate magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.



PROBLEM 6.16

IDENTIFY relevant concepts

Ampere's law:

$$\oint \vec{H} \cdot d\vec{l} = I_{f\ encl} \qquad (6.20)$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{tot\ encl}$$

Magnetic field in linear media:

$$\vec{B} = \mu_0 (1 + \chi_m) \vec{H} \qquad (6.30)$$

Bound Currents

$$\begin{cases} volume & \vec{J}_b = \vec{\nabla} \times \vec{M} \\ surface & \vec{K}_b = \vec{M} \times \hat{n} \end{cases}$$

SET UP & EXECUTE

$$\oint \vec{H} \cdot d\vec{l} = I_{f\ encl} = I \quad \Rightarrow \quad \vec{H} = \frac{I}{2\pi s} \hat{\phi}$$

PROBLEM 6.16

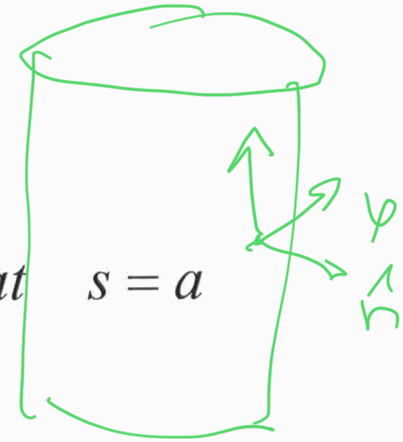
SET UP & EXECUTE

$$\vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}$$

$$\vec{M} = \chi_m \vec{H} = \frac{\chi_m I}{2\pi\, s}\hat{\phi}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(sM_\phi\right)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{z} = 0$$

$$\vec{K}_b = \vec{M}\times\hat{n} = \begin{cases} \dfrac{\chi_m I}{2\pi a}\hat{z} & at \quad s = a \\ -\dfrac{\chi_m I}{2\pi b}\hat{z} & at \quad s = b \end{cases}$$



EVALUATE

For an amperian loop between the cylinders:

$$I_{tot\ encl} = I_{f\ encl} + \int J_b da + \oint K_b dl = I + 0 + \frac{\chi_m I}{2\pi a}2\pi a = (1+\chi_m)I$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{tot\ encl} = \mu_0(1+\chi_m)I$$

$$\Rightarrow \quad \vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}$$