Problem 1.41 $\boxed{\nabla t = (\cos\theta + \sin\theta\cos\phi)\,\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\,\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}(-\sin\theta\sin\phi)\,\hat{\boldsymbol{\phi}}}$

$$\begin{aligned}
\nabla^2 t &= \nabla\cdot(\nabla t) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\cos\theta + \sin\theta\cos\phi)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(-\sin\theta + \cos\theta\cos\phi)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi) \\
&= \frac{1}{r^2}2r(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}(-2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi) - \frac{1}{r\sin\theta}\cos\phi \\
&= \frac{1}{r\sin\theta}[2\sin\theta\cos\theta + 2\sin^2\theta\cos\phi - 2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi - \cos\phi] \\
&= \frac{1}{r\sin\theta}\left[(\sin^2\theta + \cos^2\theta)\cos\phi - \cos\phi\right] = 0.
\end{aligned}$$

$\Rightarrow \boxed{\nabla^2 t = 0}$

Check: $r\cos\theta = z$, $r\sin\theta\cos\phi = x$ $\Rightarrow$ in Cartesian coordinates $t = x + z$. Obviously Laplacian is zero.

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} \nabla t\cdot d\mathbf{l} = t(\mathbf{b}) - t(\mathbf{a})$

Segment 1: $\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \to 2$. $d\mathbf{l} = dr\,\hat{\mathbf{r}}$; $\nabla t\cdot d\mathbf{l} = (\cos\theta + \sin\theta\cos\phi)dr = (0+1)dr = dr$.

$$\int \nabla t\cdot d\mathbf{l} = \int_0^2 dr = 2.$$

Segment 2: $\theta = \frac{\pi}{2}$, $r = 2$, $\phi: 0 \to \frac{\pi}{2}$. $d\mathbf{l} = r\sin\theta\, d\phi\,\hat{\boldsymbol{\phi}} = 2\,d\phi\,\hat{\boldsymbol{\phi}}$.

$$\nabla t\cdot d\mathbf{l} = (-\sin\phi)(2\,d\phi) = -2\sin\phi\,d\phi. \quad \int \nabla t\cdot d\mathbf{l} = -\int_0^{\frac{\pi}{2}} 2\sin\phi\,d\phi = 2\cos\phi\Big|_0^{\frac{\pi}{2}} = -2.$$

Segment 3: $r = 2$, $\phi = \frac{\pi}{2}$; $\theta: \frac{\pi}{2} \to 0$.

$$d\mathbf{l} = r\,d\theta\,\hat{\boldsymbol{\theta}} = 2\,d\theta\,\hat{\boldsymbol{\theta}};\quad \nabla t\cdot d\mathbf{l} = (-\sin\theta + \cos\theta\cos\phi)(2\,d\theta) = -2\sin\theta\,d\theta.$$

$$\int \nabla t\cdot d\mathbf{l} = -\int_{\frac{\pi}{2}}^{0} 2\sin\theta\,d\theta = 2\cos\theta\Big|_{\frac{\pi}{2}}^{0} = 2.$$

Total: $\int_{\mathbf{a}}^{\mathbf{b}} \nabla t\cdot d\mathbf{l} = 2 - 2 + 2 = \boxed{2}$. Meanwhile, $t(\mathbf{b}) - t(\mathbf{a}) = [2(1+0)] - [0(\ )] = 2.$ ✓

Problem 1.43

(a) $$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\,s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}\,2s(2+\sin^2\phi) + \frac{1}{s}\,s(\cos^2\phi-\sin^2\phi)+3\\ &= 4+2\sin^2\phi+\cos^2\phi-\sin^2\phi+3\\ &= 4+\sin^2\phi+\cos^2\phi+3 = \boxed{8.}\end{aligned}$$

(b) $\int(\nabla\cdot\mathbf{v})d\tau = \int(8)s\,ds\,d\phi\,dz = 8\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi\int_0^5 dz = 8(2)\left(\frac{\pi}{2}\right)(5) = \boxed{40\pi.}$

Meanwhile, the surface integral has five parts:

top: $z=5$, $d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}}$; $\mathbf{v}\cdot d\mathbf{a} = 3z\,s\,ds\,d\phi = 15s\,ds\,d\phi$. $\int\mathbf{v}\cdot d\mathbf{a} = 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi = 15\pi$.
bottom: $z=0$, $d\mathbf{a} = -s\,ds\,d\phi\,\hat{\mathbf{z}}$; $\mathbf{v}\cdot d\mathbf{a} = -3z\,s\,ds\,d\phi = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
back: $\phi=\frac{\pi}{2}$, $d\mathbf{a} = ds\,dz\,\hat{\boldsymbol{\phi}}$; $\mathbf{v}\cdot d\mathbf{a} = s\sin\phi\cos\phi\,ds\,dz = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
left: $\phi=0$, $d\mathbf{a} = -ds\,dz\,\hat{\boldsymbol{\phi}}$; $\mathbf{v}\cdot d\mathbf{a} = -s\sin\phi\cos\phi\,ds\,dz = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
front: $s=2$, $d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}}$; $\mathbf{v}\cdot d\mathbf{a} = s(2+\sin^2\phi)s\,d\phi\,dz = 4(2+\sin^2\phi)d\phi\,dz$.

$$\int\mathbf{v}\cdot d\mathbf{a} = 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)d\phi\int_0^5 dz = (4)(\pi+\tfrac{\pi}{4})(5) = 25\pi.$$

So $\oint\mathbf{v}\cdot d\mathbf{a} = 15\pi+25\pi = 40\pi$. ✓

(c) $$\begin{aligned}\nabla\times\mathbf{v} &= \left(\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right)\hat{\mathbf{s}} + \left(\frac{\partial}{\partial z}\left(s(2+\sin^2\phi)\right) - \frac{\partial}{\partial s}(3z)\right)\hat{\boldsymbol{\phi}}\\ &\quad+\frac{1}{s}\left(\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}\left(s(2+\sin^2\phi)\right)\right)\hat{\mathbf{z}}\\ &= \frac{1}{s}(2s\sin\phi\cos\phi - s\,2\sin\phi\cos\phi)\,\hat{\mathbf{z}} = \boxed{\mathbf{0}.}\end{aligned}$$

Problem 1.49

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\boldsymbol{\delta}^3(\mathbf{r})\right)\,d\tau = 4\pi e^{-0} = \boxed{4\pi.}$

Second method: integrating by parts (use Eq. 1.59).

$$J = -\int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2}\cdot\boldsymbol{\nabla}(e^{-r})\,d\tau + \oint_{\mathcal{S}} e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{a}. \quad \text{But} \quad \boldsymbol{\nabla}\left(e^{-r}\right) = \left(\frac{\partial}{\partial r}e^{-r}\right)\hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.$$

$$= \int \frac{1}{r^2}e^{-r}4\pi r^2\,dr + \int e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} = 4\pi\int_0^R e^{-r}\,dr + e^{-R}\int \sin\theta\,d\theta\,d\phi$$

$$= 4\pi\left(-e^{-r}\right)\Big|_0^R + 4\pi e^{-R} = 4\pi\left(-e^{-R} + e^{-0}\right) + 4\pi e^{-R} = 4\pi.\checkmark \quad (\text{Here } R = \infty, \text{ so } e^{-R} = 0.)$$