Problem 3.24

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}=0.$$

Look for solutions of the form $V(s,\phi)=S(s)\Phi(\phi)$:

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2}=0.$$

Multiply by s^2 and divide by $V=S\Phi$:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0.$$

Since the first term involves s only, and the second ϕ only, each is a constant:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)=C_1,\quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=C_2,\quad \text{with } C_1+C_2=0.$$

Now C_2 must be negative (else we get exponentials for Φ, which do not return to their original value—as geometrically they *must*— when ϕ is increased by 2π).

$$C_2=-k^2.\quad \text{Then } \frac{d^2\Phi}{d\phi^2}=-k^2\Phi\Rightarrow\Phi=A\cos k\phi+B\sin k\phi.$$

Moreover, since $\Phi(\phi+2\pi)=\Phi(\phi)$, *$k$ must be an integer:* $k=0,1,2,3,\ldots$ (negative integers are just repeats, but $k=0$ must be included, since $\Phi=A$ (a constant) is OK).

$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=k^2S$ can be solved by $S=s^n$, provided n is chosen right:

$$s\frac{d}{ds}\left(sns^{n-1}\right)=ns\frac{d}{ds}\left(s^n\right)=n^2ss^{n-1}=n^2s^n=k^2S\Rightarrow n=\pm k.$$

Evidently the general solution is $S(s)=Cs^k+Ds^{-k}$, *unless* $k=0$, in which case we have only *one* solution to a *second-order* equation—namely, $S=$ constant. So we must treat $k=0$ separately. *One* solution is a constant—but what's the other? Go back to the diferential equation for S, and put in $k=0$:

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=0\Rightarrow s\frac{dS}{ds}=\text{constant}=C\Rightarrow\frac{dS}{ds}=\frac{C}{s}\Rightarrow dS=C\frac{ds}{s}\Rightarrow S=C\ln s+D\ \text{(another constant)}.$$

So the second solution in this case is $\ln s$. [How about Φ? That *too* reduces to a single solution, $\Phi=A$, in the case $k=0$. What's the second solution here? Well, putting $k=0$ into the Φ equation:

$$\frac{d^2\Phi}{d\phi^2}=0\Rightarrow\frac{d\Phi}{d\phi}=\text{constant}=B\Rightarrow\Phi=B\phi+A.$$

But a term of the form $B\phi$ is unacceptable, since it does not return to its initial value when ϕ is augmented by 2π.] *Conclusion:* The general solution with cylindrical symmetry is

$$\boxed{V(s,\phi)=a_0+b_0\ln s+\sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi+b_k\sin k\phi)+s^{-k}(c_k\cos k\phi+d_k\sin k\phi)\right].}$$

Yes: the potential of a line charge goes like $\ln s$, which *is* included.

Problem 3.27 Since **r** is on the z axis, the angle α is just the polar angle θ (I'll drop the primes, for simplicity).

Monopole term:

$$\int \rho\, d\tau = kR \int \left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the r integral is

$$\int_0^R (R-2r)\,dr = \left(Rr - r^2\right)\Big|_0^R = R^2 - R^2 = 0.$$

So the monopole term is zero.

Dipole term:

$$\int r\cos\theta\rho\, d\tau = kR \int (r\cos\theta)\left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the θ integral is

$$\int_0^\pi \sin^2\theta\cos\theta\, d\theta = \frac{\sin^3\theta}{3}\Big|_0^\pi = \frac{1}{3}(0-0) = 0.$$

So the dipole contribution is likewise zero.

Quadrupole term:

$$\int r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho\, d\tau = \frac{1}{2}kR\int r^2\left(3\cos^2\theta - 1\right)\left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2\sin\theta\, dr\, d\theta\, d\phi.$$

r integral:

$$\int_0^R r^2(R-2r)\,dr = \left(\frac{r^3}{3}R - \frac{r^4}{2}\right)\Big|_0^R = \frac{R^4}{3} - \frac{R^4}{2} = -\frac{R^4}{6}.$$

θ integral:

$$\int_0^\pi \underbrace{(3\cos^2\theta - 1)}_{3(1-\sin^2\theta)-1=2-3\sin^2\theta} \sin^2\theta\, d\theta = 2\int_0^\pi \sin^2\theta\, d\theta - 3\int_0^\pi \sin^4\theta\, d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right) = \pi\left(1 - \frac{9}{8}\right) = -\frac{\pi}{8}.$$

ϕ integral:

$$\int_0^{2\pi} d\phi = 2\pi.$$

The whole integral is:

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi) = \frac{k\pi^2R^5}{48}.$$

For point P on the z axis ($r \to z$ in Eq. 3.95) the approximate potential is

$$\boxed{V(z) \cong \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}.}\quad \text{(Quadrupole.)}$$

Problem 3.29

$\mathbf{p} = (3qa - qa)\,\hat{\mathbf{z}} + (-2qa - 2q(-a))\,\hat{\mathbf{y}} = 2qa\,\hat{\mathbf{z}}$. Therefore

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

and $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, so

$$V \cong \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$