Problem 4.10

(a) $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \boxed{kR;}\ \rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -\frac{1}{r^2}3kr^2 = \boxed{-3k.}$

(b) For $r < R$, $\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}$ (Prob. 2.12), so $\mathbf{E} = \boxed{-(k/\epsilon_0)\,\mathbf{r}.}$

For $r > R$, same as if all charge at center; but $Q_{\text{tot}} = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3) = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

Problem 4.19

With no dielectric, $C_0 = A\epsilon_0/d$ (Eq. 2.54).

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D = \sigma$ between the plates. $E = \sigma/\epsilon_0$ (in air) and $E = \sigma/\epsilon$ (in dielectric). So $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$.

$$C_a = \frac{Q}{V} = \frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right) \Longrightarrow \boxed{\frac{C_a}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}.}$$

In configuration (b), with potential difference V: $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ (in air). $P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric). $\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Longrightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right).\ \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2 - 4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]

If the x axis points *down*:

	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}$
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r - 1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$

	σ_b (top surface)	σ_f (top plate)
(a)	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$
(b)	$-(\epsilon_r - 1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (left); $\frac{\epsilon_0 V}{d}$ (right)

Problem 4.20

$\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow D4\pi r^2 = \rho\frac{4}{3}\pi r^3 \Rightarrow D = \frac{1}{3}\rho r \Rightarrow \mathbf{E} = (\rho r/3\epsilon)\,\hat{\mathbf{r}}$, for $r < R$; $D4\pi r^2 = \rho\frac{4}{3}\pi R^3 \Rightarrow D = \rho R^3/3r^2 \Rightarrow \mathbf{E} = (\rho R^3/3\epsilon_0 r^2)\,\hat{\mathbf{r}}$, for $r > R$.

$$V = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = \frac{\rho R^3}{3\epsilon_0}\left.\frac{1}{r}\right|_\infty^R - \frac{\rho}{3\epsilon}\int_R^0 r\,dr = \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right).}$$