Problem 4.39

(a) Proposed potential: $\boxed{V(r) = V_0 \frac{R}{r}}$. If so, then $\boxed{\mathbf{E} = -\nabla V = V_0 \frac{R}{r^2}\,\hat{\mathbf{r}}}$, in which case $\boxed{\mathbf{P} = \epsilon_0 \chi_e V_0 \frac{R}{r^2}\,\hat{\mathbf{r}}}$, in the region $z < 0$. ($\mathbf{P} = 0$ for $z > 0$, of course.) Then $\sigma_b = \epsilon_0 \chi_e V_0 \frac{R}{R^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) = \boxed{-\frac{\epsilon_0 \chi_e V_0}{R}}$. (*Note:* $\hat{\mathbf{n}}$ points *out* of dielectric $\Rightarrow \hat{\mathbf{n}} = -\hat{\mathbf{r}}$.) This σ_b is on the surface at $r = R$. The flat surface $z = 0$ carries no bound charge, since $\hat{\mathbf{n}} = \hat{\mathbf{z}} \perp \hat{\mathbf{r}}$. Nor is there any volume bound charge (Eq. 4.39). If V is to have the required spherical symmetry, the *net* charge must be uniform:

$\sigma_{\text{tot}} 4\pi R^2 = Q_{\text{tot}} = 4\pi\epsilon_0 R V_0$ (since $V_0 = Q_{\text{tot}}/4\pi\epsilon_0 R$), so $\sigma_{\text{tot}} = \epsilon_0 V_0/R$. Therefore

$$\boxed{\sigma_f = \begin{cases} (\epsilon_0 V_0/R), & \text{on northern hemisphere} \\ (\epsilon_0 V_0/R)(1+\chi_e), & \text{on southern hemisphere} \end{cases}}.$$

(b) By construction, $\sigma_{\text{tot}} = \sigma_b + \sigma_f = \epsilon_0 V_0/R$ is uniform (on the northern hemisphere $\sigma_b = 0$, $\sigma_f = \epsilon_0 V_0/R$; on the southern hemisphere $\sigma_b = -\epsilon_0\chi_e V_0/R$, so $\sigma_f = \epsilon V_0/R$). The potential of a uniformly charged sphere is

$$V_0 = \frac{Q_{\text{tot}}}{4\pi\epsilon_0 r} = \frac{\sigma_{\text{tot}}(4\pi R^2)}{4\pi\epsilon_0 r} = \frac{\epsilon_0 V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0 \frac{R}{r}. \quad \checkmark$$

(c) Since everything is consistent, and the boundary conditions ($V = V_0$ at $r = R$, $V \to 0$ at ∞) are met, Prob. 4.38 guarantees that this is *the* solution.

(d) Figure (b) works the same way, but Fig. (a) does not: on the flat surface, $\mathbf{P}$ is *not* perpendicular to $\hat{\mathbf{n}}$, so we'd get bound charge on this surface, spoiling the symmetry.

Problem 4.27

Using Eq. 4.55: $W = \frac{\epsilon_0}{2}\int E^2\,d\tau$. From Ex. 4.2 and Eq. 3.103,

$$\mathbf{E} = \begin{cases} \dfrac{-1}{3\epsilon_0} P\,\hat{\mathbf{z}}, & (r < R) \\ \dfrac{R^3 P}{3\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), & (r > R) \end{cases}, \quad \text{so}$$

$$W_{r<R} = \frac{\epsilon_0}{2}\left(\frac{P}{3\epsilon_0}\right)^2 \frac{4}{3}\pi R^3 = \frac{2\pi}{27}\frac{P^2 R^3}{\epsilon_0}.$$

$$\begin{aligned} W_{r>R} &= \frac{\epsilon_0}{2}\left(\frac{R^3 P}{3\epsilon_0}\right)^2 \int \frac{1}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) r^2 \sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{(R^3 P)^2}{18\epsilon_0} 2\pi \int_0^\pi (1 + 3\cos^2\theta)\sin\theta\,d\theta \int_R^\infty \frac{1}{r^4}\,dr = \frac{\pi(R^3P)^2}{9\epsilon_0}\left(-\cos\theta - \cos^3\theta\right)\Big|_0^\pi \left(-\frac{1}{3r^3}\right)\Big|_R^\infty \\ &= \frac{\pi(R^3P)^2}{9\epsilon_0}\left(\frac{4}{3R^3}\right) = \frac{4\pi R^3 P^2}{27\epsilon_0}. \end{aligned}$$

$$W_{\text{tot}} = \boxed{\frac{2\pi R^3 P^2}{9\epsilon_0}}.$$

This is the correct electrostatic energy of the configuration, but it is not the "total work necessary to assemble the system," because it leaves out the mechanical energy involved in polarizing the molecules.

Using Eq. 4.58: $W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$. For $r > R$, $\mathbf{D} = \epsilon_0\mathbf{E}$, so this contribution is the same as before. For $r < R$, $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = -\frac{1}{3}\mathbf{P} + \mathbf{P} = \frac{2}{3}\mathbf{P} = -2\epsilon_0\mathbf{E}$, so $\frac{1}{2}\mathbf{D}\cdot\mathbf{E} = -2\frac{\epsilon_0}{2}E^2$, and this contribution is now $(-2)\left(\frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}\right) = -\frac{4\pi}{27}\frac{R^3P^2}{\epsilon_0}$, exactly cancelling the exterior term. *Conclusion:* $\boxed{W_{\text{tot}} = 0.}$ This is not surprising, since the derivation in Sect. 4.4.3 calculates the work done on the *free* charge, and in this problem there is no free charge in sight. Since this is a nonlinear dielectric, however, the result cannot be interpreted as the "work necessary to assemble the configuration"—the latter would depend entirely on *how* you assemble it.