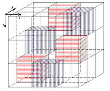


PHYSICS 604

Electricity and Magnetism
Lecture 21

Physics Department
Old Dominion University

November 30, 2021

Outline

Magnetic Field Far from Current Distribution

- Consider a localized current distribution $\mathbf{J}(\mathbf{x}')$, and the magnetic vector potential produced at a point $P(\mathbf{x})$ where $|\mathbf{x}| \gg |\mathbf{x}'|$. In the Coulomb gauge

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(x')}{|\mathbf{x}-\mathbf{x}'|}$$

- Expanding $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \ldots$, we get

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left\{\frac{1}{|\mathbf{x}|}\int d^3x'\,J_i(\mathbf{x}') + \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int d^3x'\,J_i(\mathbf{x}')\,\mathbf{x}' + \ldots\right\}$$
$$= \frac{\mu_0}{4\pi}\left\{\frac{1}{|\mathbf{x}|}\int d^3x'\,J_i(\mathbf{x}') + \frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j\int d^3x'\,J_i(\mathbf{x}')\,x'_j + \ldots\right\}$$

- We need to know the volume integrals of $J_i(\mathbf{x}')$ and $J_i(\mathbf{x}')\,x'_j$, with $J_i(\mathbf{x}')$ in principle being an arbitrary function
- For magnetostatics, however, it satisfies $\nabla\cdot\mathbf{J}(\mathbf{x}) \equiv 0$
- Integrating $\nabla'\cdot\mathbf{J}(\mathbf{x}') \equiv 0$ with any function $F(\mathbf{x}')$, we should get zero,

$$0 = -\int d^3x'\,F(\mathbf{x}')\nabla'\cdot\mathbf{J}(\mathbf{x}') = \int d^3x'\,\mathbf{J}(\mathbf{x}')\cdot\nabla' F(\mathbf{x}')$$

- In the second step we have integrated by parts and used the fact that the surface integral vanishes for a localized current distribution

Magnetic Field Far from Current Distribution, cont.

$$\Rightarrow \int d^3x' \, \mathbf{J}(\mathbf{x}') \cdot \nabla' F(\mathbf{x}') = 0$$

- In components, we have

$$\sum_{k=1}^{3} \int d^3x' J_k(\mathbf{x}') \nabla'_k F(\mathbf{x}') = 0$$

- Consider the first term in the expansion

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \left\{ \frac{1}{|\mathbf{x}|} \int d^3x' \, J_i(\mathbf{x}') + \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \int d^3x' \, J_i(\mathbf{x}') \, \mathbf{x}' + \ldots \right\}$$

- Take $F(\mathbf{x}') = x'_i$, which using $\nabla'_k x'_i = \delta_{ik}$ results in

$$\sum_{k=1}^{3} \int d^3x' \, J_k \delta_{ik} = 0 \quad \Rightarrow \quad \int d^3x' J_i = 0$$

$\Rightarrow$ the first term vanishes

- This is just a further restatement that there is no “monopole” contribution to the multipole expansion for magnetic fields

Magnetic Field Far from Current Distribution, cont.

- Consider the second term in the expansion of $\mathbf{A}(\mathbf{x})$

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left\{\frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int d^3x'\,J_i(\mathbf{x}')\,\mathbf{x}' + \ldots\right\}$$

and recall

$$\sum_{k=1}^{3}\int d^3x'\,J_k(\mathbf{x}')\nabla'_k F(\mathbf{x}') = 0$$

- To get $J_i(\mathbf{x}')\,x'_j$ in the integrand we take $F = x'_i x'_j$ and obtain

$$\begin{aligned}
\sum_{k=1}^{3}\int d^3x'\,J_k\left[\frac{\partial x'_i}{\partial x'_k}x'_j + x'_i\frac{\partial x'_j}{\partial x'_k}\right] &= 0\\
\Rightarrow \sum_{k=1}^{3}\int d^3x'\,J_k\,\left[\delta_{ik}x'_j + x'_i\delta_{jk}\right] &= 0\\
\Rightarrow \int d^3x'\,[J_i x'_j + J_j x'_i] &= 0
\end{aligned}$$

or

$$\int d^3x'\,J_i x'_j = -\int d^3x'\,J_j x'_i = \frac{1}{2}\int d^3x'\,\left[J_i x'_j - J_j x'_i\right]$$

Magnetic Field Far from Current Distribution, cont.

$$\int d^3x'\, J_i x'_j = -\int d^3x'\, J_j x'_i = \frac{1}{2}\int d^3x'\, \left[J_i x'_j - J_j x'_i\right]$$

- Thus, we may write

$$\begin{aligned} A_i(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j \int d^3x'\, J_i x'_j \\ &= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j \int d^3x' [x'_i J_j - x'_j J_i]\ , \end{aligned}$$

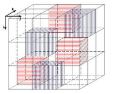
- In vector form,

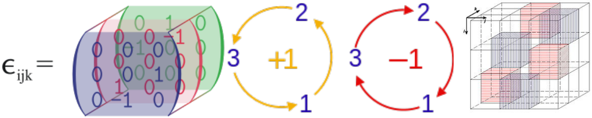
$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d^3x' [\mathbf{x}'(\mathbf{x}\cdot\mathbf{J}) - (\mathbf{x}'\cdot\mathbf{x})\mathbf{J}]\ ,$$

which may be also written as

$$\boxed{\mathbf{A}(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{x}\times\int d^3x'\, \mathbf{x}'\times\mathbf{J}}$$

using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.





- Recall the definition of the **Levi-Civita tensor**

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two of } i, j, k \text{ are equal} \\ 1 & \text{if } (ijk) \text{ is an \textit{even} permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an \textit{odd} permutation of } (123) \end{cases}$$

- This tensor is *isotropic*, and *totally anti-symmetric*. In particular, we have

$$\mathbf{A} \times \mathbf{B}|_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_j B_k \overset{\text{Einstein}}{\equiv} \epsilon_{ijk} A_j B_k$$

- There is the following identity

$$\epsilon_{ijk}\epsilon_{ilm} \equiv \sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

equivalent to $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Levi-Civita Tensor

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

- Using this identity we write

$$\begin{aligned} x'_i J_j - x'_j J_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x'_l J_m \\ &= \epsilon_{kij}\epsilon_{klm} x'_l J_m = \epsilon_{ijk}(\mathbf{x}' \times \mathbf{J})_k \end{aligned}$$

and

$$\sum_{j=1}^{3} x_j \left[x'_i J_j - x'_j J_i\right] = x_j \epsilon_{ijk}(\mathbf{x}' \times \mathbf{J})_k = \left[\mathbf{x} \times (\mathbf{x}' \times \mathbf{J})\right]_i$$

- Thus we have

$$\boxed{A_i(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\left[\mathbf{x} \times \int d^3x'\, \mathbf{x}' \times \mathbf{J}\right]_i}$$

Magnetic moment

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{x} \times \int d^3x' \, \mathbf{x}' \times \mathbf{J}$$

- **Magnetic moment** is given by the vector

$$\mathbf{m} = \frac{1}{2}\int d^3x' \, \mathbf{x}' \times \mathbf{J}$$

- **Magnetic moment density**

$$\boldsymbol{\mu} = \frac{1}{2}\mathbf{x}' \times \mathbf{J}$$

- Thus we can write

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{m} \times \mathbf{x}}$$

- This is the lowest non-vanishing term in the multipole expansion of the magnetic vector potential for a localized current density
- Applying $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{x}\cdot\mathbf{m})\mathbf{x} - r^2\mathbf{m}}{r^5}\right],$$

exactly analogous to the electrostatic field due to a point dipole

Derivation of formula for $\mathbf{B}(\mathbf{x})$

$$A_k(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\left[\mathbf{m}\times\mathbf{x}\right]_k = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\epsilon_{ijk}m_i x_j$$

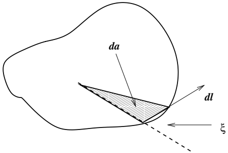
$$\left[\nabla\times(\mathbf{m}\times\mathbf{x})\frac{1}{|\mathbf{x}|^3}\right]_n = \epsilon_{nlk}\frac{\partial}{\partial x^l}\frac{1}{|\mathbf{x}|^3}\epsilon_{kij}m_i x_j$$

$$\epsilon_{nlk}\epsilon_{kij} = -\epsilon_{lnk}\epsilon_{ijk} = -\delta_{il}\delta_{jn} + \delta_{jl}\delta_{in}$$

$$\epsilon_{nlk}\epsilon_{kij}m_i x_j = (-\delta_{il}\delta_{jn} + \delta_{jl}\delta_{in})m_i x_j = -m_l x_n + m_n x_l$$

$$\frac{\partial}{\partial x^l}\frac{1}{|\mathbf{x}|^3}(-m_l x_n + m_n x_l) = 3\frac{x_l}{|\mathbf{x}|^5}(m_l x_n - m_n x_l) + \frac{1}{|\mathbf{x}|^3}(-m_l\delta_{ln} + m_n)$$
$$= 3\frac{x_n(\mathbf{x}\cdot\mathbf{m}) - m_n|\mathbf{x}|^2}{|\mathbf{x}|^5} + \frac{-m_n + 3m_n}{|\mathbf{x}|^3} = \frac{3x_n(\mathbf{x}\cdot\mathbf{m}) - m_n|\mathbf{x}|^2}{|\mathbf{x}|^5}$$

Magnetic moment of a planar loop



- For the case of a current confined to a loop, we have

$$\mathbf{m} = \frac{I}{2} \oint \mathbf{x} \times \mathbf{dl}$$

- If we have a planar loop, $\mathbf{x} \times \mathbf{dl}$ is normal to the plane of the loop, and we have

$$\frac{1}{2} \mathbf{x} \times \mathbf{dl} \quad = \quad \mathbf{n} \, \frac{1}{2} \, x \, dl \, \sin \xi = da \, \mathbf{n}$$

- As a result,

$$\mathbf{m} = I \, A \, \mathbf{n}$$

- $\mathbf{n}$ is a normal to the plane of the loop, and A is the total area of the loop

Magnetic moment of moving charged point-like particles

- Consider the case where the current distribution arises from the motion of a number of charged point-like particles:

$$\mathbf{J} = \sum_i q_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{x}_i)$$

- $\mathbf{v}_i$ is the velocity of the i^{th} particle, which we assume is much less than the velocity of light. Then we have

$$\mathbf{m} = \frac{1}{2} \sum_i q_i \mathbf{x}_i \times \mathbf{v}_i$$

- The orbital angular momentum of a particle is given by

$$\mathbf{L}_i = M_i \mathbf{x}_i \times \mathbf{v}_i$$

where M_i is the mass of the i^{th} particle. Thus, we may write

$$\mathbf{m} = \sum_i \frac{q_i}{2M_i} \mathbf{L}_i$$

- When all the particles have equal q_i/M_i ratio, the magnetic moment is proportional to the **total angular momentum**

Torques and forces on magnetic dipoles

- Consider a magnetic dipole in the uniform magnetic field $\mathbf{B}$ and take $\mathbf{m}$ due to wire loop with area a carrying current I such as $m = Ia$
- Force on a current element $I\mathbf{dl}$ at $\mathbf{x}$ in a magnetic field $\mathbf{B}(\mathbf{x})$ is

$$\mathbf{dF} = I\mathbf{dl} \times \mathbf{B}$$

- The total force acting on the loop is zero:

$$\mathbf{F} \;=\; I\oint \mathbf{dl} \times \mathbf{B} \;=\; -I\mathbf{B} \times \oint \mathbf{dl} = 0$$

- The torque acting on the loop is $\mathbf{m} \times \mathbf{B}$
  Proof: using $\mathbf{dx}' \equiv \mathbf{dl}$, we start with

$$\begin{aligned}\mathbf{N} &= \oint \mathbf{x}' \times d\mathbf{F} \;=\; \oint \mathbf{x}' \times (I\mathbf{dx}' \times \mathbf{B}) \\ &= I\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{B}) - \frac{1}{2}\mathbf{B}I\oint d(x'^2) \;=\; I\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{B})\end{aligned}$$

Torques and forces on magnetic dipoles, cont.

- For an arbitrary constant vector $\mathbf{a}$

$$\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{a}) = -\frac{1}{2}\mathbf{a} \times \oint (\mathbf{x}' \times \mathbf{dx}')$$

- Proof:

$$\mathbf{a} \times \oint (\mathbf{x}' \times \mathbf{dx}') = \oint [\mathbf{x}'(\mathbf{a} \cdot \mathbf{dx}') - \mathbf{dx}'(\mathbf{a} \cdot \mathbf{x}')] \ ,$$

furthermore

$$\oint \mathbf{x}'(\mathbf{a} \cdot \mathbf{dx}') = \oint [d(\mathbf{x}'(\mathbf{a} \cdot \mathbf{x}')) - \mathbf{dx}'(\mathbf{a} \cdot \mathbf{x}')] = -\oint \mathbf{dx}'(\mathbf{a} \cdot \mathbf{x}') \ ,$$

and therefore

$$\mathbf{a} \times \oint (\mathbf{x}' \times \mathbf{dx}') = -2 \oint \mathbf{dx}'(\mathbf{a} \cdot \mathbf{x}')$$

- Taking $\mathbf{a} = \mathbf{B}$ we get

$$\mathbf{N} = -\frac{I}{2}\mathbf{B} \times \oint (\mathbf{x}' \times \mathbf{dx}') \ = \ \left(\frac{I}{2}\oint \mathbf{x}' \times \mathbf{dx}'\right) \times \mathbf{B} \ = \ \mathbf{m} \times \mathbf{B}$$

- The torque in a uniform external field is a cross product of the magnetic moment and the field

Torques and forces on magnetic dipoles, cont.

- Consider a small dipole in the non-uniform external field (the size of the dipole $\ll$ characteristic size of the field)
- The formula for the torque remains the same: $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ (the magnetic field should be taken at the position of the dipole)
- However, the total force is no longer zero,

$$\mathbf{F} \;=\; I\oint \mathbf{dl} \times \mathbf{B} \;\neq 0$$

Since our dipole is small we can expand $\mathbf{B}(\mathbf{x}')$ in powers of $\mathbf{x}'$. Suppose that the dipole is located at the origin. We get

$$\mathbf{B}(\mathbf{x}') = \mathbf{B}(0) + (\mathbf{x}' \cdot \nabla)\,\mathbf{B}(0) + \ldots \quad \text{and using} \;\; \mathbf{dl}' \equiv \mathbf{dx}'$$

$$\mathbf{F} \;=\; I\oint \mathbf{dl}' \times \mathbf{B}(0) + I\oint \mathbf{dl}' \times (\mathbf{x}' \cdot \nabla)\mathbf{B} + O({x'}^2)$$

$$=\; I\oint \mathbf{dx}'(\mathbf{x}' \cdot \nabla) \times \mathbf{B} + O({x'}^2)$$

- Next we use

$$\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{a}) = \frac{1}{2}\oint (\mathbf{x}' \times \mathbf{dx}') \times \mathbf{a}$$

with $\mathbf{a} = \nabla$ and obtain

$$I\oint \mathbf{dx}'(\mathbf{x}' \cdot \nabla) \times \mathbf{B} \;=\; \left(\frac{I}{2}\oint (\mathbf{x}' \times \mathbf{dx}') \times \nabla\right) \times \mathbf{B} \;=\; (\mathbf{m} \times \nabla) \times \mathbf{B}$$

Torques and forces on magnetic dipoles, cont.

- Finally (incorporating $\nabla \cdot \mathbf{B} = 0$)

$$\mathbf{F} \;=\; (\mathbf{m} \times \nabla) \times \mathbf{B} \;=\; \nabla(\mathbf{m} \cdot \mathbf{B}) - \mathbf{m}(\nabla \cdot \mathbf{B}) \;=\; \nabla(\mathbf{m} \cdot \mathbf{B})$$

- Since $\mathbf{F} = -\nabla U$ we see that the potential energy of a (small) magnetic dipole in the external magnetic field is

$$U \;=\; -\mathbf{m} \cdot \mathbf{B}$$

- Relation is similar to $U = -\mathbf{p} \cdot \mathbf{E}$ for the electric dipole