# Midterm 1 solutions.



**Problem 1.** (3 points)
A parallel plate capacitor of plate separation $d$ and area $A$ is charged to voltage $V$. How much work is required to pull the plates apart so that the plates' separation becomes $2d$? Consider two cases:

(a) After initial charging to voltage $V$, the capacitor is disconnected from the battery,
and
(b) Capacitor is connected to battery so the voltage is always $V$.

**Solution** (a)
The electric field between plates is $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$ where $A$ is the area Half of this electric field is due to upper plate and half due to lower. The force acting on the upper plate of the capacitor is $F = QE_{\text{due to lower plate}} = Q\frac{\sigma}{2\epsilon_0} = \frac{Q^2}{2A\epsilon_0}$ and when you (slowly) move the plate a distance $d$ the work is

$$W = Fd = \frac{Q^2}{2A\epsilon_0}d$$

It can also be obtained from conservation of energy. Since $C = \frac{A}{d}\epsilon_0$

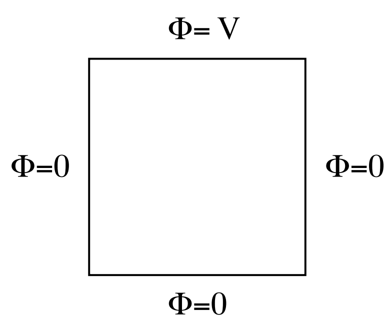
$$W = \frac{Q^2}{2C_2} - \frac{Q^2}{2C_1} = Q^2\frac{2d}{2A\epsilon_0} - Q^2\frac{d}{2A\epsilon_0} = Q^2\frac{d}{2A\epsilon_0}$$

(b)
If the distance between plates is $z$ the electric field is $E = \frac{V}{z}$ so the force is $F = \frac{QV}{2z}$ and the work to double the plates' separation is

$$W = \int_d^{2d} F(z)dz = \int_d^{2d} dz\,\frac{QV}{2z} = QV\frac{\ln 2}{2}$$

The second method is not applicable for this case since the battery is attached so the energy is not conserved.

**Problem 2.** (4 points)
Find the solution of Laplace equation in a 2-dimensional square well of size $a$ where three sides are kept at zero potential and the fourth side at constant potential $V$.




**Solution**
Similarly to Examples 3.3 and 3.4 we need to solve Laplace equation $\frac{\partial^2}{\partial x^2}V(x,y) + \frac{\partial^2}{\partial y^2}V(x,y) = 0$ with boundary conditions

$$V(0,y) = 0, \quad V(a,y) = 0, \quad V(x,0) = 0, \quad V(x,a) = V_0,$$

Separation of variables $V(x,y) = f(x)g(y)$ gives

$$\frac{\partial}{\partial x^2}f(x) = \mp k^2 f(x), \quad \frac{\partial}{\partial y^2}f(x) = \pm k^2 f(x)$$

Looking at the boundary conditions $f(0) = f(a) = 0$ we see that we need $\frac{\partial}{\partial x^2}f(x) = -k_n^2 f(x)$ with $k_n = \frac{\pi n}{a}$. Consequently, the function $g(y) = a\sinh k_n)y + b\cosh k_n y$. From boundary condition $g(0) = 0$ we get $b = 0$ so

$$f(x) = \sin\frac{\pi n}{a}x, \quad g(y) = \sinh\frac{\pi n}{a}y$$

The general solution is given by the sum over n

$$V(x,y) \;=\; \sim_{n=1}^{\infty} C_n \sin\frac{\pi n}{a} x \sinh\frac{\pi n}{a} y$$

To satisfy last boundary condition $V(x,a) = V_0$ we need

$$V(x,a) \;=\; V_0 \;=\; \sum_{n=1}^{\infty} C_n \sin\frac{\pi n}{a} x \sinh \pi n$$

Using orthogonality property (3.33) we get

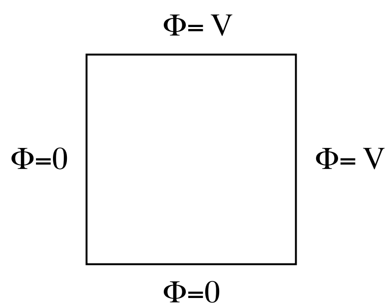
$$C_n \;=\; \frac{1}{\sinh \pi n}\frac{2V_0}{a}\int_0^a dx\ \sin\frac{\pi n x}{a} \;=\; \frac{2V}{n \sinh \pi n}\left[1-(-1)^n\right]$$

and the solution takes the form

$$V(x,y) \;=\; \frac{4V_0}{\pi}\sum_{n=\mathrm{odd}} \frac{1}{n \sinh \pi n} \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a} \tag{1}$$

*Extra credit - 1 points.*

Same for setup shown below




**Solution**

The result for the setup with $V = 0$ at $y = 0, a$, $x = 0$ and $V = V_0$ at $x = a$ is the same with trivial substitution $x \leftrightarrow y$

$$V(x,y) \;=\; \frac{4V}{\pi}\sum_{n=\mathrm{odd}} \frac{1}{n \sinh \pi n} \sin\frac{\pi n y}{a} \sinh\frac{\pi n x}{a} \tag{2}$$

By superposition principle, the result for $V = 0$ at $y = 0$, $x = 0$ and $V = V_0$ at $x = a$, $y = a$ is the sum of Eqs. (1) and (2)

$$V(x,y) \;=\; \frac{4V}{\pi}\sum_{n=\mathrm{odd}} \frac{1}{n \sinh \pi n} \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a} \;+\; (x \leftrightarrow y)$$

**Problem 3**. (3 points).
A pure dipole is located at distance $d$ from the infinite conducting plane and is aligned with the normal to the plane. What is the force between the dipole and the conducting plane? Is it attractive or repulsive?

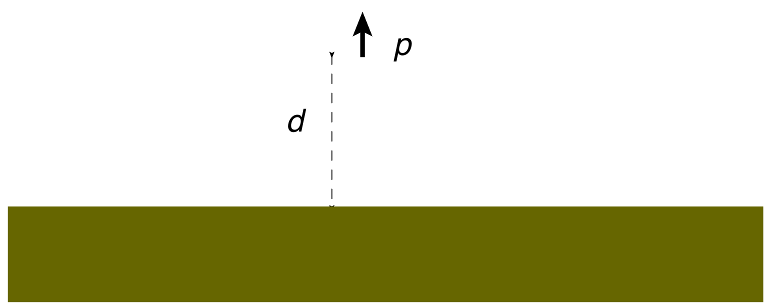
**Solution**
It is easy to see that the image dipole is equal to the original dipole and is located at distance $-d$. Let us assume that the conducting plane lies at $z = 0$. The field due to the image dipole art the point $z, 0, 0$ is

$$\vec{E} = \frac{3(\vec{p}\cdot\hat{e}_3)\hat{e}_3 - \vec{p}}{|z+d|^3} = \frac{2p}{(z+d)^3}\hat{e}_3$$

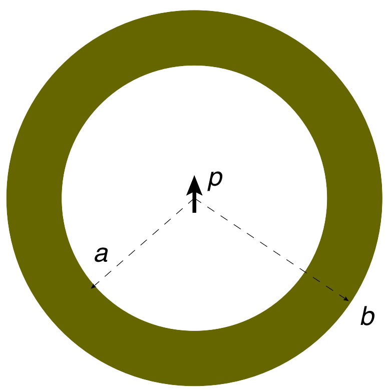
and the force on the original diple due to image dipole is

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} = p\frac{\partial}{\partial z}E_z\Big|_{z=d}\hat{e}_3 = p\frac{\partial}{\partial z}\frac{2p}{(z+d)^3}\Big|_{z=d}\hat{e}_3 = -\frac{3p^2}{4d^3}\hat{e}_3$$

The force is attractive




**Problem 4**. A pure dipole is located at the center of the conducting spherical shell of inner radius $a$ and outer radius $b$.




(a) What is the electric field outside the shell? (2 points)
(b) What is the potential inside (at $r < a$) ? (4 points)

**Solution**
(a)
From the solution of Example (2.10)

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} = 0$$

since the total charge inside the cavity $Q = 0$.

(b)
Expansion in Legendre polynomials has the form

$$V(r,\theta) \;=\; \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$$

Since we have a dipole inside with the potential $\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$ we should allow the term with $B_1$

$$V(r,\theta) \;=\; \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + B_1 r^{-2}\cos\theta$$

From the behavior at $r \to 0$ we see that $B_1 = \frac{p}{4\epsilon_0}$. We know that $V(a,\theta) = 0$ so

$$V(a,\theta) \;=\; \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + B_1 r^{-2}\cos\theta \;=\; 0$$

From the orthogonality of Legendre polynomials we see that $A_i = 0$ if $l \neq 1$ and $A_1 = \frac{B_1}{a^3}$ so

$$V(r,\theta) \;=\; \frac{p\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{a^3}\right]$$

*Extra credit - 1 point.* What method you would use if the dipole is not at the center?
If dipole is not at the center, the problem can be solved by method of images.