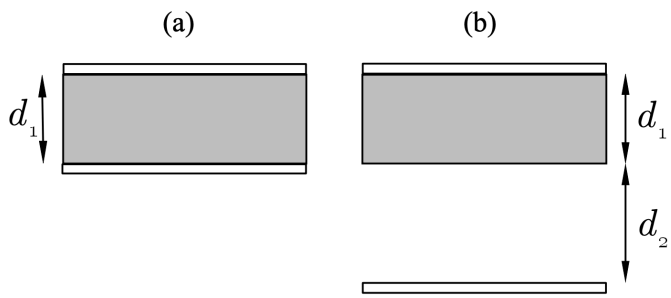
Phys 425 midterm #1 (16 points).

Problem 1. (4 points)
A parallel plate capacitor of plate separation d_1 is filled with a solid dielectric material of permittivity ϵ, as shown in the figure (a) below. The capacitor is charged to voltage V_1. The capacitor is then disconnected from the battery and pulled apart so that the plate separation becomes $d_1 + d_2$. The dielectric does not expand and the dielectric-free region has size d_2 (see figure (b)]. Assume the plates are large compared to both $d_1$and d_2.
Find:
(I) the voltage V_2 after the capacitor is pulled apart.
(II) the surface charge density at the lower surface of the dielectric in part (b)



Solution
(i)The surface charge density is

$$\sigma \;=\; \frac{C_1 V_0}{A} \;=\; \frac{\epsilon}{d_1} V_0$$

For the part (b) the electric displacement in the dielectric $D_1 = \sigma$ so the electric field is $E_1 = \frac{\sigma}{\epsilon}$. The electric field in the dielectric-free region is $E_2 = \frac{\sigma}{\epsilon_0}$ so the total voltage is

$$V_2 \;=\; \frac{\sigma}{\epsilon} d_1 + \frac{\sigma}{\epsilon_0} d_2 \;=\; \frac{Q}{\frac{A}{\epsilon} d_1} + \frac{Q}{\frac{A}{\epsilon_0} d_2} \;=\; V_0 \Big(1 + \frac{\epsilon_0 d_2}{\epsilon d_1} \Big)$$

which corresponds to the capacitors with $C_1 \;=\; \frac{A}{\epsilon} d_1$ and $C_2 \;=\; \frac{A}{\epsilon_0} d_2$ connected in series $\frac{1}{C} \;=\; \frac{1}{C_1} + \frac{1}{C_2}$.
(ii). The surface charge density on the lower surface of dielectric is

$$\sigma \;=\; \big(E_{\perp}^{\text{above}} - E_{\perp}^{\text{below}} \big) \epsilon_0 \;=\; -\,\sigma \frac{\epsilon - \epsilon_0}{\epsilon}$$

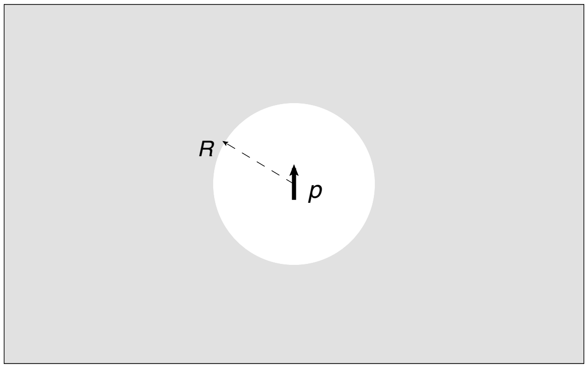
Problem 2. (4 points)
Dielectric material with permittivity ϵ fills the whole space except for the spherical cavity with radius R. A pure dipole p is placed in the center of the cavity. Find the potential inside and outside the cavity.

Solution
A general expansion in Legendre polynomials has the form

$$V(r, \theta) \;=\; \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

At $r > R$

$$V_{\text{out}}(r, \theta) \;=\; \sum_{l=0}^{\infty} B_l r^{-l-1}) P_l(\cos\theta)$$



Since we have a dipole inside with the potential $\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$ we should allow the term with b_1 inside

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) + b_1 r^{-2}\cos\theta$$

From the behavior at $r \to 0$ we see that $b_1 = \frac{p}{4\pi\epsilon_0}$. We know that $V(R,\theta)$ is continuous so

$$\sum_{l=0}^{\infty} B_l R^{-l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta) + b_1 R^{-2}\cos\theta$$

Next, there is no free charge on the surface so $D_\perp = D_r$ is continuous $\Leftrightarrow \epsilon \frac{\partial V}{\partial r}\Big|_{r\to R+} = \epsilon_0 \frac{\partial V}{\partial r}\Big|_{r\to R-}$ and we get

$$\frac{\epsilon}{\epsilon_0}\sum_{l=0}^{\infty}(-(l+1)B_l)R^{-l-2}P_l(\cos\theta) = \sum_{l=0}^{\infty} l a_l R^{l-1} P_l(\cos\theta) - 2b_1 R^{-3}\cos\theta$$

From the orthogonality of Legendre polynomials we see that $a_i = b_i = B_i = 0$ if $l \neq 1$ and

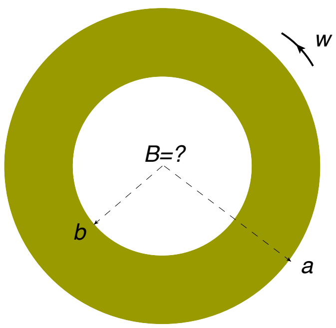
$$B_1 = a_1 R^3 + b_1, \qquad -2\frac{\epsilon}{\epsilon_0}B_1 = a_1 R^3 - 2b_1 \qquad \Rightarrow \qquad a_1 = \frac{2(\epsilon_0 - \epsilon)}{2\epsilon + \epsilon_0}\frac{p}{4\pi\epsilon_0}, \qquad B_1 = \frac{3\epsilon_0}{2\epsilon + \epsilon_0}\frac{p}{4\pi\epsilon_0}$$

and therefore

$$V_{\text{out}}(r,\theta) = \frac{3\epsilon_0}{2\epsilon + \epsilon_0}\frac{p\cos\theta}{4\pi\epsilon_0 r^2}, \qquad V_{\text{in}}(r,\theta) = \frac{2(\epsilon_0 - \epsilon)}{2\epsilon + \epsilon_0}\frac{pR\cos\theta}{4\pi\epsilon_0 R^3} + \frac{p\cos\theta}{4\pi\epsilon_0 r^2},$$

Problem 3. (4 points)

A disc of radius a has a hole of radius b whose center is located at the center of a larger disc. The charge Q is spread uniformly over this disc. The disc is rotating with angular velocity ω around axis perpendicular to the surface of the disc and passing through its center. Find the magnetic field at the center of the hole.



Solution #1

The magnetic field in the center of a circular wire of radius r carrying current I is

$$\vec{B} \;=\; \frac{\mu_0 I}{2\pi r}\hat{e}_3$$

(Eq. (5.41) from *Griffiths* at $z=0$). A stripe of the disc between r and $r+dr$ has current $dI \;=\; \sigma v dr \;=\; \omega\sigma r dr$ so it creates

$$d\vec{B} \;=\; \frac{\mu_0 dI}{2\pi r}\hat{e}_3 \;=\; \frac{\mu_0 \sigma\omega}{2\pi}\hat{e}_3 dr$$

and therefore

$$\vec{B} \;=\; \frac{\mu_0 \omega\sigma}{2\pi}\hat{e}_3 \int_b^a dr \;=\; \frac{\mu_0\omega\sigma(a-b)}{2\pi}\hat{e}_3$$

Solution #2

Formula (5.42) from *Griffiths*

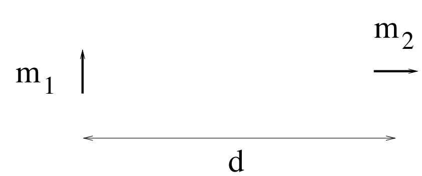
$$\vec{B} \;=\; \frac{\mu_0}{4\pi}\int da' \; \vec{K}(\vec{r'}) \times \frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3} \;\stackrel{\vec{r}=0}{=}\; -\frac{\mu_0}{4\pi}\int da' \; \vec{K}(\vec{r'}) \times \frac{\vec{r'}}{|\vec{r'}|^3}$$

For the rotating disc $\vec{K} = \sigma\vec{v} = \sigma\vec{\omega}\times\vec{r'}$ so

$$\vec{B} \;=\; \frac{\mu_0}{4\pi}\int da' \;\;=\;\; -\frac{\mu_0\sigma}{4\pi}\int da' \; (\vec{\omega}\times\vec{r'})\times\frac{\vec{r'}}{|\vec{r'}|^3} \;=\; \frac{\mu_0\sigma\vec{\omega}}{4\pi}\int\frac{da'}{r'} \;=\; \frac{\mu_0\sigma\vec{\omega}}{2}\int_b^a dr' \;=\; \frac{\mu_0\omega\sigma(a-b)}{2\pi}\hat{e}_3$$

Problem 4 (4 points).

Two pure magnetic dipoles $\vec{m}_1 = m\hat{e}_1$ and $\vec{m}_2 = m_2\hat{e}_3$ are separated by distance $\vec{d} = d\hat{e}_3$.



Find

a) Energy of this setup.

b) Force between the dipoles.

Solution

The potential energy for two dipoles separated by $\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$ is

$$U \;=\; -(\vec{m}_2\cdot\vec{B}_{(1)}(\vec{r}) \;=\; \frac{\mu_0}{4\pi r^4}[(\vec{m}_1\cdot\vec{m}_2)r^2 - 3(\vec{m}_1\cdot\vec{r})(\vec{m}_2\cdot\vec{r})] \;=\; -\frac{\mu_0}{4\pi}\frac{3m_1m_2xz}{(x^2+y^2+z^2)^2}$$

$\Rightarrow$ the potential energy at the point $\vec{r} = d\hat{e}_3$ is 0 and the force is

$$\vec{F} \;=\; -\vec{\nabla}U \;=\; -\frac{\mu_0}{4\pi}\frac{3m_1m_2z}{(x^2+y^2+z^2)^2}\hat{e}_1 \;=\; -\frac{\mu_0}{4\pi}\frac{3m_1m_2z}{d^3}\hat{e}_1$$