454 Midterm 1 (16 points).

Problem 1.

You store 10.0 kg of ice at -30.0°C in a perfectly thermally insulating vessel. You then open it, quickly pour 2.0 kg of water at +50.0°C on the ice, and immediately seal the vessel.

(a) What will the temperature be when the system has reached thermal equilibrium? [Ignore thermal effects of air.]

(b)How much ice (in kg) will there be in the end?

Specific heat of water: 4.184 kJ/(kg K)

Specific heat of ice: 2.06 kJ/(kg K)

Latent heat of fusion of water: 334.2 kJ/kg

Solution.

We have 3 possibilities: all ice melt, all water freeze, or ice and water coexist at 0°C.

Consider them in turn:

1. Maximal amount of heat released due to cooling of the water is $4.184 kJ/(kgK) \times 50°K \times 2kg = 418.4$kJ which is smaller than the heat $334.2 \times 10kJ = 3342kJ$ necessary to melt all ice (even disregarding the fact that ice needs to warm up from -30°K).

2. Maximal amount of heat absorbed due to warming of the ice is $2.06 kJ/(kgK) \times 30°K \times 10kg = 618$kJ which is smaller than the heat $334.2(kJ/kg) \times 2kg = 668.4$kJ necessary to freeze all water (even disregarding the fact that water should first cool down to 0°K).

3. Thus, the final temperature is 0°C and some water has frozen or some ice melted. Denote mass of the new water m_x (it can be negative). We get

$$c_1 m_1 \Delta T_1 - c_2 m_2 \Delta T_2 = l m_x$$
$$\Rightarrow m_x = \frac{c_1 m_1 \Delta T_1 - c_2 m_2 \Delta T_2}{l} = \frac{4.184 \times 2 \times 50 - 2.06 \times 10 \times 30}{334.2} = \frac{418.4 - 618}{334.2} \simeq -0.6\text{kg}$$

which means that there will be 0.6 kg of new ice. Overall, there will be 10.6 kg of ice and 1.4 kg of water at 0°C.

Problem 2.

A heat engine operates on an ideal monoatomic gas in the reversible cycle consisting of isochoric, isobaric and two isothermal processes as shown below. What is the efficiency of this heat engine?

Solution.

The internal energy for an ideal gas is $u = c_v T$, and for the monoatomic gas $c_v = \frac{3}{2}R$.
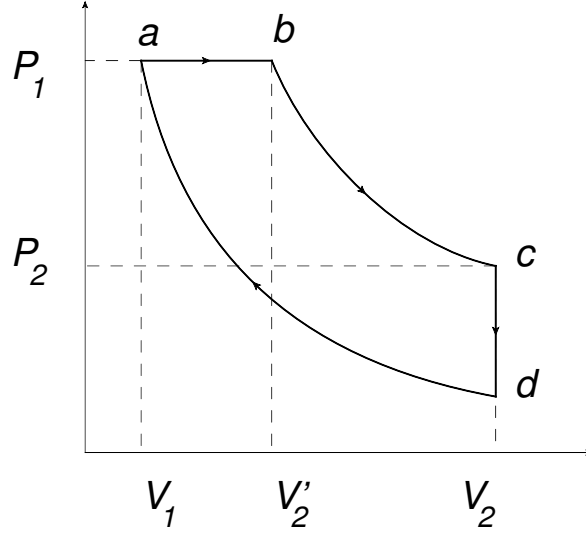
FIG. 1. PV diagram for a heat engine. Curved lines are isotherms.

Let us assume positive sign for heat going into the device and consider four processes of the cycle.

    1. Process a → b.

$$P_1 V_2' = P_2 V_2 \quad \Rightarrow \quad W_{ab} \; = \; P_1(V_2' - V_1) \; = \; P_2 V_2 - P_1 V_1 \; = \; nR(T_2 - T_1)$$

$$U_b - U_a \; = \; C_v(T_2 - T_1) \quad \Rightarrow \quad Q_{ab} \; = \; n(c_v + R)(T_2 - T_1) \; = \; n\frac{5R}{2}(T_2 - T_1)$$

    2. Process b → c.

$$U_b = U_c \quad \Rightarrow \quad Q_{bc} \; = \; W_{bc} \; = \; nRT_2 \ln\frac{V_2}{V_2'} \; = \; nRT_2 \ln\frac{P_1}{P_2}$$

    3. Process c→d.

$$W_{cd} \; = \; 0 \quad \Rightarrow \quad Q_{cd} \; = \; U_d - U_c \; = \; nc_v(T_1 - T_2)$$

    4. Process d→a.

$$U_d = U_a \quad \Rightarrow \quad Q_{da} \; = \; W_{da} \; = \; nRT_1 \ln\frac{V_1}{V_2}$$

    Total work:

$$W \; = \; W_{ab} + W_{bc} + W_{da} \; = \; nR(T_2 - T_1) + nRT_2 \ln\frac{P_1}{P_2} - nRT_1 \ln\frac{V_2}{V_1}$$

Total heat absorbed:

$$Q_{ab} + Q_{bc} \; = \; \frac{5}{2}nR(T_2 - T_1) + nRT_2 \ln\frac{P_1}{P_2}$$

Check: total heat rejected $|Q_{cd}| + |Q_{da}| = \frac{3}{2}nR(T_2 - T_1) + nRT_1 \ln \frac{V_2}{V_1}$ so

$$Q_{ab} + Q_{bc} - |Q_{cd}| - |Q_{da}| = W \qquad \text{OK}$$

Efficiency:

$$\eta = \frac{W}{Q_{ab} + Q_{bc}} = \frac{nR(T_2 - T_1) + nRT_2 \ln \frac{P_1}{P_2} - nRT_1 \ln \frac{V_2}{V_1}}{\frac{5}{2}nR(T_2 - T_1) + nRT_2 \ln \frac{P_1}{P_2}} = \frac{T_2 - T_1 + T_2 \ln \frac{P_1}{P_2} - T_1 \ln \frac{V_2}{V_1}}{\frac{5}{2}(T_2 - T_1) + T_2 \ln \frac{P_1}{P_2}}$$

Compare to Carnot efficiency

$$\frac{T_2 - T_1 + T_2 \ln \frac{P_1}{P_2} - T_1 \ln \frac{V_2}{V_1}}{\frac{5}{2}(T_2 - T_1) + T_2 \ln \frac{P_1}{P_2}} < 1 - \frac{T_1}{T_2}$$

since

$$\frac{5}{2}\frac{(T_2 - T_1)^2}{T_2} + T_1 \ln \frac{P_2 V_2}{P_1 V_1} = \frac{5}{2}\frac{(T_2 - T_1)^2}{T_2} + T_1 \ln \frac{T_2}{T_1} > T_2 - T_1$$

Problem 3.

A kilomole of ideas gas undergoes a reversible isothermal expansion from volume V to volume $3V$.

(a) What is the change in entropy of the gas?

(b) What is the change of entropy of the universe?

Suppose now that the same expansion takes place as a free expansion. Same questions:

(c) What is the change in entropy of the gas?

(d) What is the change of entropy of the universe?

Solution.

(a):
Change in entropy of the gas in the isothermal process

$$\Delta S_{\text{gas}} = \frac{\Delta Q}{T}, \quad \Delta Q = W = nRT \ln \frac{V_2}{V_1} \Rightarrow \Delta S_{\text{gas}} = nR \ln 3$$

(b):
Change of the entropy of the surroundings

$$\Delta S_{\text{surr}} = -\frac{\Delta Q}{T}, \quad \Rightarrow \Delta S_{\text{surr}} = -nR \ln 3$$

so the entropy of the universe does not change as it should be for a reversible process.

(c):

Change of the entropy of the gas is the same as in the reversible process:

$$\Delta S_{\text{gas}} \; = \; nRT \ln \frac{V_2}{V_1} \quad \Rightarrow \quad \Delta S_{\text{gas}} \; = \; nR \ln 3$$

(d):

Entropy of the surroundings does not change $\Rightarrow$ the entropy of the universe is increased by $nR \ln 3$.