603 final (40 points). 5/2/17, 12:30-3:30 p.m.

Problem 1.

A rocket with velocity v_∞ and impact parameter b approaches the planet with mass M and radius R. What is the condition that the rocket will hit the planet?

Solution

Solution is similar to problem 1 from the midterm. The effective potential at $r > R$ is

$$V_{\text{eff}} = -G\frac{Mm}{r} + \frac{\mu v_\infty^2 b^2}{2r^2}$$

where $\mu = \frac{Mm}{M+m}$ is the effective mass. The condition that the rocket hits the planet is

$$E = \frac{\mu}{2}v_\infty^2 \geq -G\frac{Mm}{R} + \frac{\mu v_\infty^2 b^2}{2R^2} \quad \Leftrightarrow \quad \frac{\mu}{2}v_\infty^2\big(\frac{b^2}{R^2} - 1\big) \leq G\frac{Mm}{R}$$

Since mass of the rocket is much smaller than the mass of a planet, $\mu \simeq m$ and the condition reads

$$v_\infty^2\big(\frac{b^2}{R^2} - 1\big) \leq \frac{2M}{R}G$$

Problem 2.

Problem 2.4 from the textbook (Fetter and Walecka)

Solution

Similarly to Sect. 2.3.1 from the lecture notes we solve the equation

$$m\ddot{\vec{r}} = mg - 2m\vec{\omega} \times \dot{\vec{r}}$$

in the first two orders in ω:

$$\vec{r}(t) = \vec{r}^{(0)}(t) + \vec{r}^{(1)}(t)$$

(the correction to g due to centrifugal force is $\sim \omega^2$). We choose a local frame on the earth's surface with $\hat{e}_x$ southward, $\hat{e}_y$ eastward, and $\hat{e}_z$ vertically upward as shown in Fig. 1. In the leading order we get

$$\ddot{\vec{r}} = \vec{g} = -g\hat{e}_z$$

so

$$\vec{r}^{(0)}(t) = \big(v_0 t - \frac{1}{2}gt^2\big)\hat{z}$$

The maximum height is $h = \frac{v_0^2}{2g}$ so $v_0 = \sqrt{2gh}$.

In the first two orders in ω we get

$$\ddot{\vec{r}}^{(0)} + \ddot{\vec{r}}^{(1)} = -g\hat{z} - 2\vec{\omega} \times \dot{\vec{r}}^{(0)} \quad \Rightarrow \quad \ddot{\vec{r}}_1(t) = 2\vec{\omega} \times \dot{\vec{r}}_0(t) = 2\vec{\omega} \times \hat{z}(v_0 - gt)$$

From Fig. 1 we see that $\omega \times \hat{z} = -\omega\hat{y}\sin\theta$ so

$$\dot{\vec{r}}^{(1)}(t) = -\omega\big(2v_0 t - gt^2\big)\hat{e}_y \quad \Rightarrow \quad \vec{r}^{(1)}(t) = -\omega\big(v_0 t^2 - \frac{gt^3}{3}\big)\hat{e}_y$$

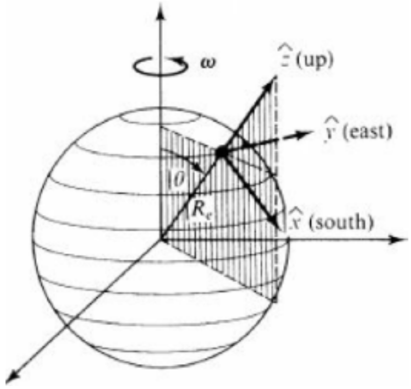


Figure 1. Earth-fixed frame

(recall that $\vec{r}^{(1)}(0) = \dot{\vec{r}}^{(1)}(0) = 0$) and the total trajectory becomes

$$\vec{r}(t) \;=\; \big(v_0 t - \frac{g}{2}t^2\big)\hat{e}_z \;-\; \omega\big(v_0 t^2 - \frac{gt^3}{3}\big)\hat{e}_y$$

The particle falls back on earth at $t = \frac{2v_0}{g} = \sqrt{\frac{8h}{g}}$ so the deflection is westward:

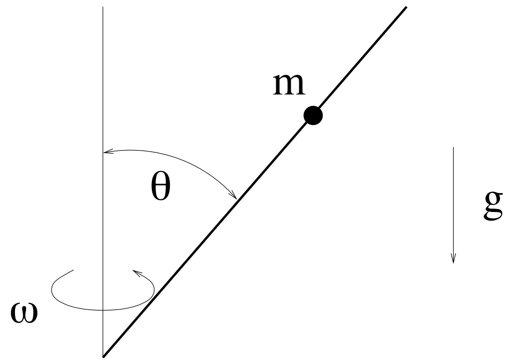
$$\Delta\vec{r} \;=\; -\,\omega\big(\frac{4v_0^3}{g^2} - \frac{8v_0^3}{3g^2}\big)\hat{e}_y \sin\theta \;=\; -\,\frac{4v_0^3\omega}{3g^2}\sin\theta\hat{e}_y \;=\; -\,\frac{8\omega}{3}\sqrt{\frac{2h^3}{g}}\sin\theta\hat{e}_y$$

The deflection is the upper point is also westward and two times smaller (as evident from symmetry of up and down motion)):

$$\omega\Big[v_0\big(\frac{v_0}{g}\big)^2 - \frac{g}{3}\big(\frac{v_0}{g}\big)^3\Big]\hat{e}_y \;=\; -\,\frac{2v_0^3\omega}{3g^2}\sin\theta\hat{e}_y \;=\; -\,\frac{4\omega}{3}\sqrt{\frac{2h^3}{g}}\sin\theta\hat{e}_y$$

Problem 3.

A bead of mass m slides without friction along a straight wire that is rotating with constant angular frequency $\dot{\phi} = \omega$ about a vertical axis. The wire makes a fixed angle θ with the rotation axis. Gravity acts downward.



1. Construct the Lagrangian of the bead in a suitable generalized coordinate(s).

2. Obtain Euler-Lagrange equation(s) and find the condition for an equilibrium circular orbit of radius $s_0 \sin\theta$

3. Is this equilibrium stable or unstable? Explain.

Solution

Lagrangian

$$L = \frac{m}{2}(\dot{s}^2 + s^2\omega^2 \sin^2\theta) - mgs\cos\theta$$

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = m\ddot{s} = \frac{\partial L}{\partial s} = m\omega^2 s \sin^2\theta - mg\cos\theta$$

Equilibrium $\ddot{s} = 0 \;\Rightarrow$

$$\omega^2 s_0 \sin^2\theta = g\cos\theta \quad\Rightarrow\quad s_0 = \frac{g\cos\theta}{\omega^2\sin^2\theta}$$

Stable or unstable: $s(t) = s_0 + \sigma(t)$

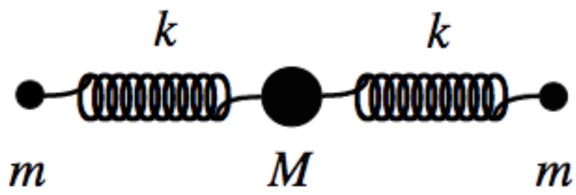
$$L = \frac{m}{2}[\dot{\sigma}^2 + (s_0+\sigma)^2\omega^2\sin^2\theta] - mg(s_0+\sigma)\cos\theta = \frac{m}{2}(\dot{\sigma}^2 + \sigma^2\omega^2\sin^2\theta) + \text{const}$$

The sign of potential energy is opposite to harmonic oscillator $\Rightarrow$ the equilibrium is unstable.

Problem 4.

A linear symmetric molecule consists of three atoms on one straight line as shown in the figure below. The forces between molecules are modeled by two identical springs of spring constant k. Consider only variations along the line of the molecules.

1. Write the Lagrangian and Euler-Lagrange equations for this system.
2. Find the characteristic frequencies of the molecule's oscillations.



Solution

The Lagrangian is

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{M}{2}\dot{x}_3^2 + \frac{k}{2}(x_{13}^2 + x_{23}^2)$$

Position of c.m.

$$x = \frac{m(x_1 + x_2) + Mx_3}{2m + M}$$

Distances from the c.m.

$$y_1 = x_1 - x = \frac{mx_{12} + Mx_{13}}{2m + M}, \quad y_2 = x_2 - x = \frac{-mx_{12} + Mx_{23}}{2m + M}, \quad y_3 = x_3 - x = -\frac{m(x_{13} + x_{23})}{2m + M}$$

Check: $m(y_1 + y_2) + My_3 = 0$

Generalized coordinates: x, y_1, y_2. Since $y_3 = -\frac{m}{M}(y_1 + y_2)$

$$L = \frac{m}{2}\left[(\dot{x} + \dot{y}_1)^2 + (\dot{x} + \dot{y}_2)^2\right] + \frac{M}{2}\left[\dot{x} - \frac{m}{M}(\dot{y}_1 + \dot{y}_2)\right]^2 - \frac{k}{2}\left(\left[y_1 + \frac{m}{M}(y_1 + y_2)\right]^2 + \left[y_2 + \frac{m}{M}(y_1 + y_2)\right]^2\right)$$
$$= \left(m + \frac{M}{2}\right)\dot{x}^2 + \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2 + \frac{m}{M}(\dot{y}_1 + \dot{y}_2)^2\right] - \frac{k}{2}\left[y_1^2 + y_2^2 + \frac{2m^2 + 2Mm}{M^2}(y_1 + y_2)^2\right]$$

Motion of the c.m. is irrelevant so L can be reduced to

$$L = \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2 + \frac{m}{M}(\dot{y}_1 + \dot{y}_2)^2\right] - \frac{k}{2}\left[y_1^2 + y_2^2 + \frac{2m^2 + 2Mm}{M^2}(y_1 + y_2)^2\right]$$

Let us introduce new variables

$$z_1 \equiv y_1 + y_2, \qquad z_2 \equiv y_1 - y_2$$

The Lagrangian takes the form

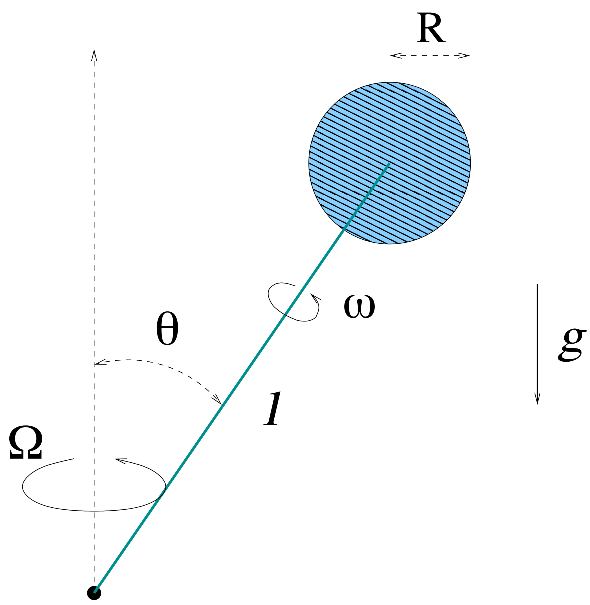
$$
\begin{aligned}
L &= \frac{m}{2}\Big[\Big(\frac{1}{2}+\frac{m}{M}\Big)\dot{z}_1^2+\frac{1}{2}\dot{z}_2^2\Big]-\frac{k}{2}\Big[\frac{(2m+M)^2}{2M^2}z_1^2+\frac{1}{2}z_2^2\Big] \;=\; L_1+L_2 \\
L_1 &= \frac{m}{4}\Big(1+\frac{2m}{M}\Big)\dot{z}_1^2-\frac{k}{4}\frac{(2m+M)^2}{M^2}, \qquad L_2 \;=\; \frac{m}{4}\dot{z}_2^2-\frac{k}{4}z_2^2
\end{aligned}
$$

Thus, z_1 and z_2 are normal modes with characteristic frequencies

$$
\omega_1 \;=\; \sqrt{\frac{k}{m}}\sqrt{1+\frac{2m}{M}}, \qquad \omega_2 \;=\; \sqrt{\frac{k}{m}}
$$

Problem 5.

A gyroscope is made from a sphere with mass m and radius R and a rod of length l and negligible mass. It is rotating around its axle with constant angular velocity ω. The axle is pivoted at its end without friction. The gravity acts downward.



Assuming that the resulting precession about z axis is completely uniform so that the nutation effects are absent, express the angular velocity of precession Ω in terms of ω and parameters of the gyroscope.

Solution

The last line in Eq. (5.67) from lecture notes reads

$$I_1\ddot{\beta} \;=\; \frac{\partial L}{\partial \beta} \;=\; I_1\dot{\alpha}^2\sin\beta\cos\beta - I_3\dot{\alpha}\sin\beta(\dot{\alpha}\cos\beta + \dot{\gamma}) + mgl\sin\beta$$

For a steady circular precession motion the l.h.s. vanishes so

$$\frac{\partial L}{\partial \beta} \;=\; I_1\dot{\alpha}^2\sin\beta\cos\beta - I_3\dot{\alpha}\sin\beta(\dot{\alpha}\cos\beta + \dot{\gamma}) + mgl\sin\beta \;=\; 0$$

In addition, for the sphere $I_1 = I_3 = I = \frac{2}{5}mR^2$ so one gets

$$I\dot{\alpha}\dot{\gamma}\sin\beta \;=\; mgl\sin\beta \quad\Rightarrow\quad \Omega \;\equiv\; \dot{\alpha} \;=\; \frac{mgl}{I\dot{\gamma}} \;=\; \frac{5}{2}\frac{gl}{R^2\omega}$$

Problem 6.

Consider the Hamiltonian

$$H \;=\; \frac{p^2}{2m} + \frac{\vec{a}\cdot\vec{r}}{r^3}$$

where p is the magnitude of vector $\vec{p}$, r is the magnitude of vector $\vec{r}$ and $\vec{a}$ is an arbitrary constant vector.

1. Evaluate the Poisson bracket $[\vec{r}\cdot\vec{p}, H]$
2. Given $\vec{r}_0$ and $\vec{p}_0$ at time $t = 0$, find $\vec{r}\cdot\vec{p}$ at a later time t.

Solution

Since the potential energy does not depend on time, the energy (Hamiltonian) is conserved:

$$H(t) \;=\; H \;=\; \frac{p_0^2}{2m} + \frac{\vec{a}\cdot\vec{r}_0}{r_0^3}$$

The Poisson bracket is

$$[\vec{r}\cdot\vec{p}, H] \;=\; \sum_{i=1}^{3}\Big(\frac{\partial\vec{r}\cdot\vec{p}}{\partial r_i}\frac{\partial H}{\partial p_i} - \frac{\partial\vec{r}\cdot\vec{p}}{\partial p_i}\frac{\partial H}{\partial r_i}\Big) \;=\; \frac{p^2}{m} + 2\frac{\vec{a}\cdot\vec{r}}{r^3} \;=\; 2H$$

where I used

$$\frac{\partial H}{\partial r_i} \;=\; \frac{a_i}{r^3} - \frac{3r_i}{r^5}\vec{a}\cdot\vec{r}$$

Since

$$\frac{d}{dt}\vec{r}\cdot\vec{p} \;=\; [\vec{r}\cdot\vec{p}, H] \;=\; 2H$$

and H does not depend on time,

$$\vec{r}\cdot\vec{p} \;=\; \vec{r}_0\cdot\vec{p}_0 \;+\; Ht \;=\; \vec{r}_0\cdot\vec{p}_0 \;+\; \Big(\frac{p_0^2}{2m} + \frac{\vec{a}\cdot\vec{r}_0}{r_0^3}\Big)t$$