

603 final (40 points). 5/2/17, 12:30-3:30 p.m.

Problem 1.

A rocket with velocity v_∞ and impact parameter b approaches the planet with mass M and radius R . What is the condition that the rocket will hit the planet?

Solution

Solution is similar to problem 1 from the midterm. The effective potential at $r > R$ is

$$V_{\text{eff}} = -G \frac{Mm}{r} + \frac{\mu v_\infty^2 b^2}{2r^2}$$

where $\mu = \frac{Mm}{M+m}$ is the effective mass. The condition that the rocket hits the planet is

$$E = \frac{\mu}{2} v_\infty^2 \geq -G \frac{Mm}{R} + \frac{\mu v_\infty^2 b^2}{2R^2} \Leftrightarrow \frac{\mu}{2} v_\infty^2 \left(\frac{b^2}{R^2} - 1 \right) \leq G \frac{Mm}{R}$$

Since mass of the rocket is much smaller than the mass of a planet, $\mu \simeq m$ and the condition reads

$$v_\infty^2 \left(\frac{b^2}{R^2} - 1 \right) \leq \frac{2M}{R} G$$

Problem 2.

Problem 2.4 from the textbook (Fetter and Walecka)

Solution

Similarly to Sect. 2.3.1 from the lecture notes we solve the equation

$$m \ddot{\vec{r}} = m\vec{g} - 2m\vec{\omega} \times \dot{\vec{r}}$$

in the first two orders in ω :

$$\vec{r}(t) = \vec{r}^{(0)}(t) + \vec{r}^{(1)}(t)$$

(the correction to g due to centrifugal force is $\sim \omega^2$). We choose a local frame on the earth's surface with \hat{e}_x southward, \hat{e}_y eastward, and \hat{e}_z vertically upward as shown in Fig. 1. In the leading order we get

$$\ddot{\vec{r}} = \vec{g} = -g\hat{e}_z$$

so

$$\vec{r}^{(0)}(t) = (v_0 t - \frac{1}{2} g t^2) \hat{z}$$

The maximum height is $h = \frac{v_0^2}{2g}$ so $v_0 = \sqrt{2gh}$.

In the first two orders in ω we get

$$\ddot{\vec{r}}^{(0)} + \ddot{\vec{r}}^{(1)} = -g\hat{z} - 2\vec{\omega} \times \dot{\vec{r}}^{(0)} \Rightarrow \ddot{\vec{r}}_1(t) = 2\vec{\omega} \times \dot{\vec{r}}_0(t) = 2\vec{\omega} \times \hat{z}(v_0 - gt)$$

From Fig. 1 we see that $\omega \times \hat{z} = -\omega \hat{y} \sin \theta$ so

$$\dot{\vec{r}}^{(1)}(t) = -\omega(2v_0 t - gt^2) \hat{e}_y \Rightarrow \vec{r}^{(1)}(t) = -\omega \left(v_0 t^2 - \frac{gt^3}{3} \right) \hat{e}_y$$

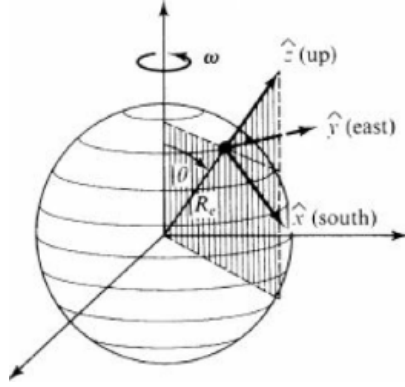


Figure 1. Earth-fixed frame

(recall that $\vec{r}^{(1)}(0) = \dot{\vec{r}}^{(1)}(0) = 0$) and the total trajectory becomes

$$\vec{r}(t) = \left(v_0 t - \frac{g}{2} t^2\right) \hat{e}_z - \omega \left(v_0 t^2 - \frac{g t^3}{3}\right) \hat{e}_y$$

The particle falls back on earth at $t = \frac{2v_0}{g} = \sqrt{\frac{8h}{g}}$ so the deflection is westward:

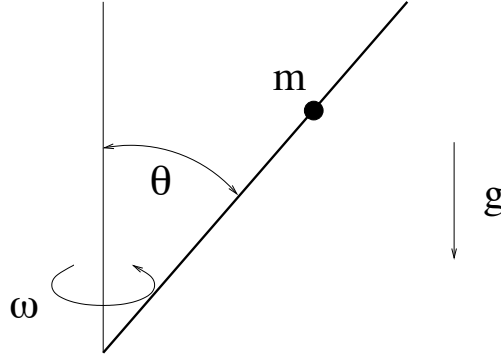
$$\Delta \vec{r} = -\omega \left(\frac{4v_0^3}{g^2} - \frac{8v_0^3}{3g^2}\right) \hat{e}_y \sin \theta = -\frac{4v_0^3 \omega}{3g^2} \sin \theta \hat{e}_y = -\frac{8\omega}{3} \sqrt{\frac{2h^3}{g}} \sin \theta \hat{e}_y$$

The deflection at the upper point is also westward and two times smaller (as evident from symmetry of up and down motion):

$$\omega \left[v_0 \left(\frac{v_0}{g}\right)^2 - \frac{g}{3} \left(\frac{v_0}{g}\right)^3 \right] \hat{e}_y = -\frac{2v_0^3 \omega}{3g^2} \sin \theta \hat{e}_y = -\frac{4\omega}{3} \sqrt{\frac{2h^3}{g}} \sin \theta \hat{e}_y$$

Problem 3.

A bead of mass m slides without friction along a straight wire that is rotating with constant angular frequency $\dot{\phi} = \omega$ about a vertical axis. The wire makes a fixed angle θ with the rotation axis. Gravity acts downward.



1. Construct the Lagrangian of the bead in a suitable generalized coordinate(s).
2. Obtain Euler-Lagrange equation(s) and find the condition for an equilibrium circular orbit of radius $s_0 \sin \theta$
3. Is this equilibrium stable or unstable? Explain.

Solution

Lagrangian

$$L = \frac{m}{2}(\dot{s}^2 + s^2\omega^2 \sin^2 \theta) - mgs \cos \theta$$

Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = m\ddot{s} = \frac{\partial L}{\partial s} = m\omega^2 s \sin^2 \theta - mg \cos \theta$$

Equilibrium $\ddot{s} = 0 \Rightarrow$

$$\omega^2 s_0 \sin^2 \theta = g \cos \theta \quad \Rightarrow \quad s_0 = \frac{g \cos \theta}{\omega^2 \sin^2 \theta}$$

Stable or unstable: $s(t) = s_0 + \sigma(t)$

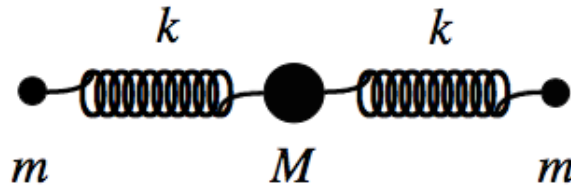
$$L = \frac{m}{2}[\dot{\sigma}^2 + (s_0 + \sigma)^2\omega^2 \sin^2 \theta] - mg(s_0 + \sigma) \cos \theta = \frac{m}{2}(\dot{\sigma}^2 + \sigma^2\omega^2 \sin^2 \theta) + \text{const}$$

The sign of potential energy is opposite to harmonic oscillator \Rightarrow the equilibrium is unstable.

Problem 4.

A linear symmetric molecule consists of three atoms on one straight line as shown in the figure below. The forces between molecules are modeled by two identical springs of spring constant k . Consider only variations along the line of the molecules.

1. Write the Lagrangian and Euler-Lagrange equations for this system.
2. Find the characteristic frequencies of the molecule's oscillations.



Solution

The Lagrangian is

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{M}{2}\dot{x}_3^2 + \frac{k}{2}(x_{13}^2 + x_{23}^2)$$

Position of c.m.

$$x = \frac{m(x_1 + x_2) + Mx_3}{2m + M}$$

Distances from the c.m.

$$y_1 = x_1 - x = \frac{mx_{12} + Mx_{13}}{2m + M}, \quad y_2 = x_2 - x = \frac{-mx_{12} + Mx_{23}}{2m + M}, \quad y_3 = x_3 - x = -\frac{m(x_{13} + x_{23})}{2m + M}$$

Check: $m(y_1 + y_2) + My_3 = 0$

Generalized coordinates: x, y_1, y_2 . Since $y_3 = -\frac{m}{M}(y_1 + y_2)$

$$\begin{aligned} L &= \frac{m}{2}[(\dot{x} + \dot{y}_1)^2 + (\dot{x} + \dot{y}_2)^2] + \frac{M}{2}[\dot{x} - \frac{m}{M}(\dot{y}_1 + \dot{y}_2)]^2 - \frac{k}{2}\left(\left[y_1 + \frac{m}{M}(y_1 + y_2)\right]^2 + \left[y_2 + \frac{m}{M}(y_1 + y_2)\right]^2\right) \\ &= \left(m + \frac{M}{2}\right)\dot{x}^2 + \frac{m}{2}[\dot{y}_1^2 + \dot{y}_2^2 + \frac{m}{M}(\dot{y}_1 + \dot{y}_2)^2] - \frac{k}{2}\left[y_1^2 + y_2^2 + \frac{2m^2 + 2Mm}{M^2}(y_1 + y_2)^2\right] \end{aligned}$$

Motion of the c.m. is irrelevant so L can be reduced to

$$L = \frac{m}{2}[\dot{y}_1^2 + \dot{y}_2^2 + \frac{m}{M}(\dot{y}_1 + \dot{y}_2)^2] - \frac{k}{2}\left[y_1^2 + y_2^2 + \frac{2m^2 + 2Mm}{M^2}(y_1 + y_2)^2\right]$$

Let us introduce new variables

$$z_1 \equiv y_1 + y_2, \quad z_2 \equiv y_1 - y_2$$

The Lagrangian takes the form

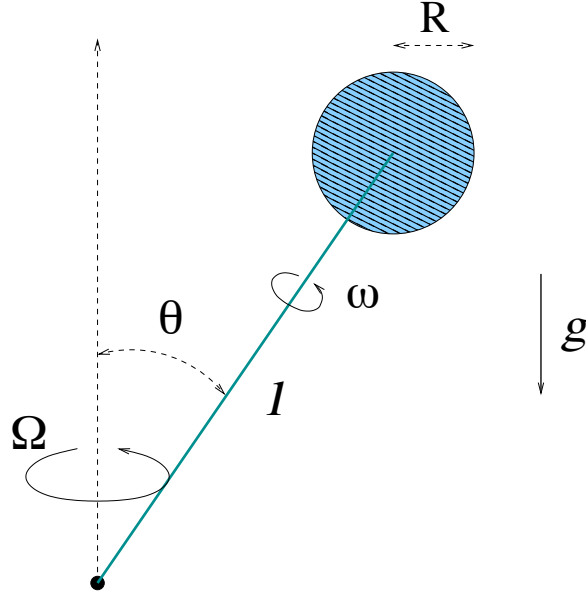
$$L = \frac{m}{2} \left[\left(\frac{1}{2} + \frac{m}{M} \right) \dot{z}_1^2 + \frac{1}{2} \dot{z}_2^2 \right] - \frac{k}{2} \left[\frac{(2m+M)^2}{2M^2} z_1^2 + \frac{1}{2} z_2^2 \right] = L_1 + L_2$$
$$L_1 = \frac{m}{4} \left(1 + \frac{2m}{M} \right) \dot{z}_1^2 - \frac{k}{4} \frac{(2m+M)^2}{M^2} z_1^2, \quad L_2 = \frac{m}{4} \dot{z}_2^2 - \frac{k}{4} z_2^2$$

Thus, z_1 and z_2 are normal modes with characteristic frequencies

$$\omega_1 = \sqrt{\frac{k}{m}} \sqrt{1 + \frac{2m}{M}}, \quad \omega_2 = \sqrt{\frac{k}{m}}$$

Problem 5.

A gyroscope is made from a sphere with mass m and radius R and a rod of length l and negligible mass. It is rotating around its axle with constant angular velocity ω . The axle is pivoted at its end without friction. The gravity acts downward.



Assuming that the resulting precession about z axis is completely uniform so that the nutation effects are absent, express the angular velocity of precession Ω in terms of ω and parameters of the gyroscope.

Solution

The last line in Eq. (5.67) from lecture notes reads

$$I_1 \ddot{\beta} = \frac{\partial L}{\partial \beta} = I_1 \dot{\alpha}^2 \sin \beta \cos \beta - I_3 \dot{\alpha} \sin \beta (\dot{\alpha} \cos \beta + \dot{\gamma}) + mgl \sin \beta$$

For a steady circular precession motion the l.h.s. vanishes so

$$\frac{\partial L}{\partial \beta} = I_1 \dot{\alpha}^2 \sin \beta \cos \beta - I_3 \dot{\alpha} \sin \beta (\dot{\alpha} \cos \beta + \dot{\gamma}) + mgl \sin \beta = 0$$

In addition, for the sphere $I_1 = I_3 = I = \frac{2}{5}mR^2$ so one gets

$$I \dot{\alpha} \dot{\gamma} \sin \beta = mgl \sin \beta \quad \Rightarrow \quad \Omega \equiv \dot{\alpha} = \frac{mgl}{I \dot{\gamma}} = \frac{5}{2} \frac{gl}{R^2 \omega}$$

Problem 6.

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\vec{a} \cdot \vec{r}}{r^3}$$

where p is the magnitude of vector \vec{p} , r is the magnitude of vector \vec{r} and \vec{a} is an arbitrary constant vector.

1. Evaluate the Poisson bracket $[\vec{r} \cdot \vec{p}, H]$
2. Given \vec{r}_0 and \vec{p}_0 at time $t = 0$, find $\vec{r} \cdot \vec{p}$ at a later time t .

Solution

Since the potential energy does not depend on time, the energy (Hamiltonian) is conserved:

$$H(t) = H = \frac{p_0^2}{2m} + \frac{\vec{a} \cdot \vec{r}_0}{r_0^3}$$

The Poisson bracket is

$$[\vec{r} \cdot \vec{p}, H] = \sum_{i=1}^3 \left(\frac{\partial \vec{r} \cdot \vec{p}}{\partial r_i} \frac{\partial H}{\partial p_i} - \frac{\partial \vec{r} \cdot \vec{p}}{\partial p_i} \frac{\partial H}{\partial r_i} \right) = \frac{p^2}{m} + 2 \frac{\vec{a} \cdot \vec{r}}{r^3} = 2H$$

where I used

$$\frac{\partial H}{\partial r_i} = \frac{a_i}{r^3} - \frac{3r_i}{r^5} \vec{a} \cdot \vec{r}$$

Since

$$\frac{d}{dt} \vec{r} \cdot \vec{p} = [\vec{r} \cdot \vec{p}, H] = 2H$$

and H does not depend on time,

$$\vec{r} \cdot \vec{p} = \vec{r}_0 \cdot \vec{p}_0 + Ht = \vec{r}_0 \cdot \vec{p}_0 + \left(\frac{p_0^2}{2m} + \frac{\vec{a} \cdot \vec{r}_0}{r_0^3} \right) t$$