

Phys. 807 — Statistical Mechanics

Solution.

(a)

Suppose the wheel moves with velocity v to the right and consider the point on the rim which was passing origin at $t = 0$. At time t the wheel rotates at the angle $\theta = \omega t = \frac{v}{a}t$ clockwise so in wheel's frame the coordinates of the point would be

$$x' = -a \sin \omega t, \quad y' = a(1 - \cos \omega t)$$

However, during time t the center of the wheel shifts to the right at $x_0 = vt = \omega a$ so the coordinates of the point in rest frame are

$$x = x' + \omega t = x' = -\omega t - a \sin \omega t, \quad y' = a(1 - \cos \omega t)$$

Eliminating t in favor of $\theta = \omega t$ we get

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta)$$

The cycloid turned upside down is

$$x = a(\theta - \sin \theta), \quad y = a(-1 + \cos \theta) = -2a \sin^2 \frac{\theta}{2} \quad (1)$$

Next, using

$$\begin{aligned} dx &= a(1 - \cos \theta)d\theta = 2a \sin^2 \frac{\theta}{2} d\theta \\ dy &= -a \sin \theta = -2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \end{aligned}$$

we get the element of the length of the cycloid

$$ds = \sqrt{dx^2 + dy^2} = 2a \sin \frac{\theta}{2} d\theta$$

so the distance measured along the cycloid from the lowest point is

$$s = \int_0^s ds' = 2a \int_0^\theta d\theta' \sin \frac{\theta'}{2} = 4a \cos \frac{\theta}{2} \quad (2)$$

(b)

The equation for brachistochrone is (3.78) from the lecture notes

$$y'' = -\frac{1+y'^2}{2y}$$

From Eq. (2) we get

$$\begin{aligned} y' &= \frac{dy}{dx} = -\cot \frac{\theta}{2} \\ y'' &\equiv \frac{d}{dx} y' = -\frac{d}{dx} \cot \frac{\theta}{2} = \frac{\frac{d\theta}{dx}}{2 \sin^2 \frac{\theta}{2}} = \frac{1}{2 \sin^2 \frac{\theta}{2} \frac{dx}{d\theta}} = \frac{1}{4a \sin^4 \frac{\theta}{2}} \end{aligned} \quad (3)$$

and it is easy to see that our $y(x)$ from Eq. (1) and $y'(x)$ and $y''(x)$ from Eq. (3) satisfy the above brachistochrone equation.

(c)

Using s from Eq. (2) as a generalized coordinate we get the Lagrangian in the form

$$L = \frac{m}{2} \dot{s}^2 - mgy = \frac{m}{2} \dot{s}^2 + 2amg \sin^2 \frac{\theta}{2} = \frac{m}{2} \dot{s}^2 + 2amg \left(1 - \frac{s^2}{16a^2}\right)$$

The constant $2amg$ is just a shift of potential energy so we can use

$$L = \frac{m}{2} \dot{s}^2 - mg \frac{s^2}{8a}$$

instead. This is the Lagrangian for harmonic oscillator with $\omega = \sqrt{\frac{g}{4a}}$