HW assignment 7.

A particle of mass m and charge q is constrained to move along a frictionless circular wire of radius R located in the vertical plane. Another charge q is attached to the lowest point of the wire. The system is in the gravitational field with gravitational acceleration g pointing downwards. Find:
a) The Lagrangian
b) Equilibrium position
c) Frequency of small oscillations around the equilibrium.

Solution

(a)
The Lagrangian is

$$L = \frac{m}{2}R^2\dot{\theta}^2 - V(\theta) = \frac{m}{2}R^2\dot{\theta}^2 + mgR\cos\theta - \frac{q^2/4\pi\epsilon_0}{2R\sin\frac{\theta}{2}}$$

$\Rightarrow$ Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \qquad \Rightarrow \qquad mR^2\ddot{\theta} = -V'(\theta) = -mgR\sin\theta + \frac{q^2}{16\pi\epsilon_0 R}\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

We will need also

$$-V''(\theta) = \frac{\partial}{\partial\theta}\Big[-mgR\sin\theta + \frac{q^2}{16\pi\epsilon_0 R}\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\Big] = -mgR\cos\theta + \frac{q^2}{16\pi\epsilon_0 R}\Big(\frac{1}{2\sin\frac{\theta}{2}} - \frac{1}{\sin^3\frac{\theta}{2}}\Big)$$

At the equilibrium

$$mgR\sin\theta = \frac{q^2}{16\pi\epsilon_0 R}\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

Let us denote $s = \left(\frac{q^2}{32\pi\epsilon_0 mgR^2}\right)^{\frac{1}{3}}$, then:
If $s < 1$ the stable equilibrium is at

$$\sin\frac{\phi_0}{2} = s$$

and the frequency of small oscillations is

$$\omega_0^2 = \frac{3g}{R}(1-s^2)$$

There is also an unstable (at $s < 1$) equilibrium at $\theta = \pi$ (the highest point). However, when $s \geq 1$ this equilibrium is stable and the frequency of small oscillations is

$$\omega_0^2 = \frac{g}{R}(s^3 - 1)$$