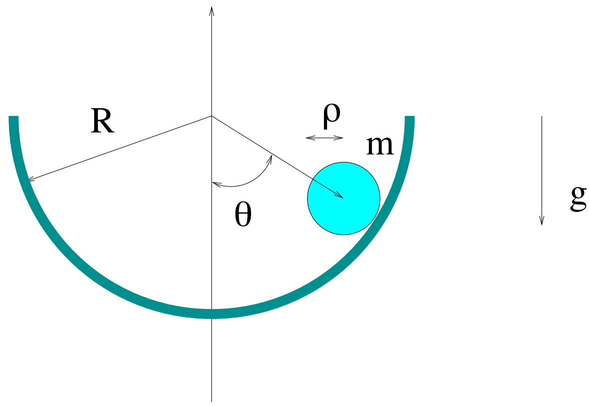
HW assignment 8.

A sphere of radius ρ and mass m is constrained to roll without slipping on a lower half of the inner surface of the hollow, stationary cylinder of inside radius R as shown in the figure below.
Find the Lagrangian for the sphere.



Solution
Let us choose the angle θ and the coordinate z along the cylinder's axis as generalized coordinates. Eq. (1.40) from the lecture notes:

$$T = \frac{mv^2}{2} + T_{\text{c.m}} = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{m}{2}\dot{\theta}^2(R-\rho)^2 + \frac{2}{5}m\rho^2\dot{\phi}^2$$

First part is easy:

$$v^2 = \frac{m}{2}(R-\rho^2)\dot{\theta}^2 + \frac{m}{2}\dot{z}^2$$

To find ω is a little bit more tricky. If the sphere moved an infinitesimal distance ds without slipping it turned on the angle $d\phi = \frac{ds}{\rho}$. There are two components in $\vec{ds}$: $ds_z = dz$ in z direction and $ds_\theta = Rd\theta$ in the x, y plane along the cylinder. They are mutually orthogonal so

$$ds = \sqrt{dz^2 + R^2 d\theta^2} \quad \Rightarrow \quad d\phi = \frac{ds}{\rho} = \sqrt{\frac{dz^2}{\rho^2} + \frac{R^2}{\rho^2}d\theta^2}$$

and the magnitude angular velocity of the sphere is

$$\omega = \frac{d\phi}{dt} = \sqrt{\frac{\dot{z}^2}{\rho^2} + \frac{R^2}{\rho^2}\dot{\theta}^2}$$

(the direction is irrelevant for our purpose since for the sphere $I_1 = I_2 = I_3 = \frac{2}{5}m\rho^2$ and the kinetic energy in c.m. frame is $\frac{I\omega^2}{2}$).

$$T = \frac{m}{2}\dot{\theta}^2(R-\rho)^2 + \frac{m}{5}R^2\dot{\theta}^2$$

Thus, the kinetic energy is

$$T = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{m}{2}[(R-\rho)^2\dot{\theta}^2 + \dot{z}^2] + \frac{2}{5}m\rho^2\left(\frac{\dot{z}^2}{\rho^2} + \frac{R^2}{\rho^2}\dot{\theta}^2\right)$$

The potential energy is $-mg(R-\rho)\cos\theta$ so

$$L = \frac{m}{2}\left[(R-\rho)^2 + \frac{2}{5}R^2\right]\dot{\theta}^2 + \frac{7}{5}\frac{m\dot{z}^2}{2} + mg(R-\rho)\cos\theta$$