Contents

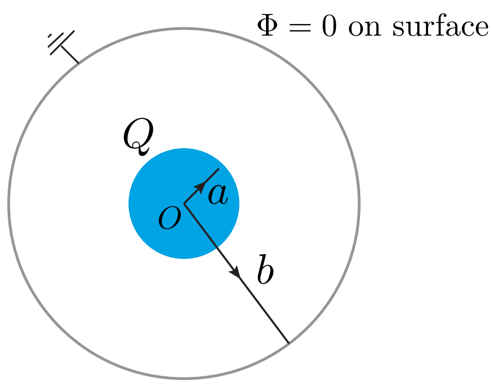
Chapter 2

Boundary-Value Problems in Electrostatics

In this chapter we will examine solutions to Poisson's and Laplace's equations in electrostatics. Before we proceed to a formal solution of Poisson's equation, we will look at a few simple solutions. In the next section we will exploit the *uniqueness theorem* in a particularly neat way through the *Method of Images*, but first, back to Gauss' Law for a simple example...

2.1 Preliminaries

Example: Charged sphere inside grounded, conducting shell.
A sphere of radius a, carrying a spherically symmetric charge distribution with the total charge Q, is placed inside a grounded, conducting sphere of radius b ($b > a$). Find the potential in the region $a \leq r \leq b$.



Thus we have to solve Poisson's equation, subject to the boundary conditions $\Phi(\mathbf{r}) = 0$ for

$r = b$. Apply Gauss' Law to the region $a < r < b$:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\ \mathbf{e_r}\ ; \qquad a \le r \le b \tag{2.1.1}$$

for which the potential is

$$\Phi = \frac{Q}{4\pi\epsilon_0 r} + \Phi_0\ ; \qquad a \le r \le b\ , \tag{2.1.2}$$

where Φ_0 is a constant.
The boundary conditions tell us that Φ vanishes at $r = b$. Thus we have

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right); \qquad a \le r \le b\ . \tag{2.1.3}$$

Let us check that our solution for $\Phi(\mathbf{r})$ satisfies Poission's equation for $a \le r \le b$. We are implicitly working in spherical polars $(r,\ \theta,\ \varphi)$, therefore (from your favourite vector-calculus course, or back of *Jackson*):

$$\begin{aligned}
\nabla^2\Phi(r,\theta,\varphi) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial^2\Phi}{\partial\varphi^2}\right\} \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(-\frac{1}{r^2}\right)\right\} \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(-1\Big) \ = \ 0
\end{aligned} \tag{2.1.4}$$

Hence $\Phi(\mathbf{r})$ satisfies $\nabla^2\Phi(\mathbf{r}) = 0$ in the charge free region $a \le r \le b$, and satisfies the boundary condition $\Phi(b) = 0$ on the surface. Therefore, *it is the unique solution of Poisson's equation in this region.* Of course, due to spherical symmetry, $\Phi(\mathbf{r})$ doesn't depend on θ or φ, and therefore the calculation of the $\nabla^2\Phi(\mathbf{r})$ is particularly simple.
Finally, let us find the surface charge density on the conductor. At the boundary of the conductor,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 b^2}\,\mathbf{e_r} = -\frac{Q}{4\pi\epsilon_0 b^2}\,\hat{\mathbf{n}}\ , \tag{2.1.5}$$

where $\hat{\mathbf{n}}$ is the normal to the conductor surface, oriented *outward*. Thus the surface charge density is given by

$$\sigma = -\frac{Q}{4\pi b^2} \tag{2.1.6}$$

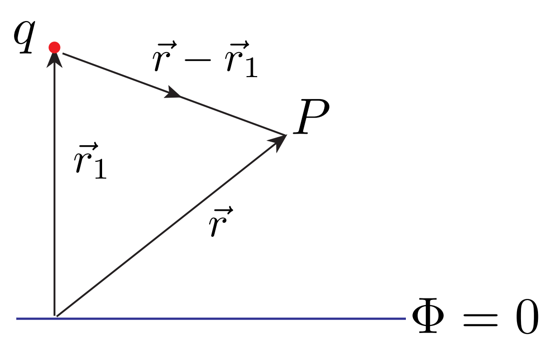
which has negative sign compared to Q, as expected. Indeed the total induced charge on the conductor is equal and opposite to that of the charge distribution.

Once again, the method was particularly simple in this case because of *spherical symmetry.* Similar simplifications occur in the case of *cylindrical symmetry.*

2.2 Method of Images

The uniqueness property of the solutions of Laplace's and Poisson's Equations leads to a neat method of obtaining their solution in particular geometric cases.

Consider a charge q placed at $\mathbf{r_1} = h\mathbf{k}$ above an infinite grounded conducting plane at $z = 0$, as shown on the right. Then on the conducting plane the potential must vanish. Thus, in the space above the $z = 0$ plane, we have the Poisson's equation

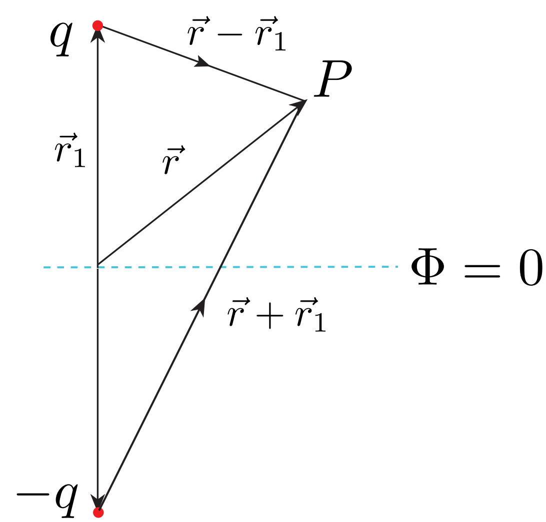


$$\nabla^2_{\mathbf{r}}\Phi(\mathbf{r}) = -4\pi q\delta^3(\mathbf{r} - \mathbf{r_1}) \ , \tag{2.2.1}$$

with the boundary condition

$$\Phi(\mathbf{r})|_{z=0} = 0 \ . \tag{2.2.2}$$

Now consider a system with a charge q placed at $\mathbf{r_1}$, and a charge $-q$ placed at $-\mathbf{r_1}$ in the absence of the conducting plane, as shown on the right. The potential $\Phi(\mathbf{r}) \equiv \Phi_1(\mathbf{r}) + \Phi_2(\mathbf{r})$ is



$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - \mathbf{r_1}|} + \frac{-q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} + \mathbf{r_1}|}. \tag{2.2.3}$$

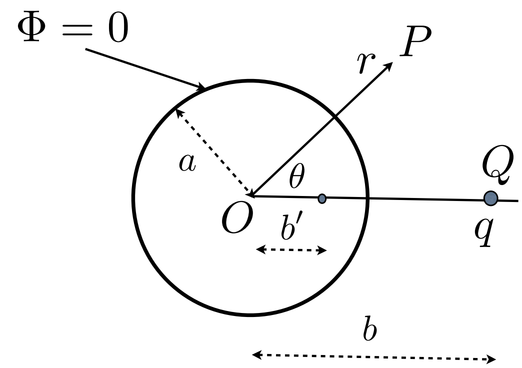
At $z = 0$, the potential vanishes because here points are equidistant from the positive and negative charges. Thus, we have $\Phi(\mathbf{r})|_{z=0} = 0$. Furthermore, the potential $\Phi(\mathbf{r})$ satisfies the Poisson's equation

$$\nabla^2_{\mathbf{r}}\Phi(\mathbf{r}) = -4\pi q\left[\delta^3(\mathbf{r} - \mathbf{r_1}) - \delta^3(\mathbf{r} + \mathbf{r_1})\right] \ . \tag{2.2.4}$$

In the space *above* the plane $z = 0$, it coincides with Eq. (2.2.2) since $\delta^3(\mathbf{r} + \mathbf{r_1})$ vanishes there. In other words, for $z > 0$, we have Poisson's equation for a point charge at $\mathbf{r_1}$, since no further changes have been introduced in this region (the only charge we have introduced is *below* the plane $z = 0$). Thus, by our uniqueness theorem, the potential **above** $z = 0$ **plane** is the same as that of a charge q placed above a grounded sheet at $z = 0$.

2.2.1 Point Charge near grounded Sphere

Consider a point charge q placed at a distance b from the center of a grounded conducting sphere of radius $a < b$. We will now show that an equivalent problem is to place an *image charge* $q' = -qa/b$ as shown, at a distance $b' = a^2/b$ from the center of the sphere.



By symmetry, the image charge q' must lie along OQ, at a distance b', say, from the center of the sphere. Thus the resultant potential of the image system is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} + \frac{q'}{|\mathbf{r}-\mathbf{b}'|}\right\}. \tag{2.2.5}$$

We need two equations to determine q' and b'; we will obtain these by imposing that Φ vanishes at the two points where OQ intersects the sphere

$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{b-a} + \frac{q'}{a-b'}\right\} &= 0 \\ \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{a+b} + \frac{q'}{a+b'}\right\} &= 0. \end{aligned}$$

For the ratio q'/q, we obtain

$$\frac{q'}{q} = -\frac{a-b'}{b-a} = -\frac{a+b'}{a+b}\ . \tag{2.2.6}$$

If $A/B = C/D \equiv \alpha$, then $(A+C)/(B+D) = (\alpha B + \alpha D)/(B+D) = \alpha$. Hence, $A/B = C/D = (A+C)/(B+D)$ and we have $q'/q = -a/b$ or

$$q' = -q\,\frac{a}{b}\ . \tag{2.2.7}$$

Now, from $q'/q = (-a)/b$ and $q'/q = (a-b')/(a-b)$ we have $q'/q = (-b')/a$ and

$$q'/q = (-a)/b = (-b')/a \quad \Rightarrow \quad b' = a^2/b\ . \tag{2.2.8}$$

Finally, let us verify that Φ does indeed vanish for *all* points on the surface of the sphere.

On the surface,

$$\begin{aligned} |\mathbf{r}-\mathbf{b}'|^2 &= a^2 - 2a\frac{a^2}{b}\cos\theta + \frac{a^4}{b^2} \\ &= \frac{a^2}{b^2}\left\{a^2 - 2ab\cos\theta + b^2\right\} \\ &= \frac{a^2}{b^2}|\mathbf{r}-\mathbf{b}|^2 \,, \end{aligned} \tag{2.2.9}$$

and hence

$$\Phi(\mathbf{r})|_{r=a} = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} - \frac{qa}{b}\frac{1}{|\mathbf{r}-\mathbf{b}|a/b}\right\} = 0. \tag{2.2.10}$$

Thus we have

1. The image system satisfies the original Poisson's equation for $r \geq a$ since the only additional charge we have introduced is in the region $r < a$.

2. The potential for the image system satisfies the condition $\Phi = 0$ at $r = a$.

Thus, by the uniqueness theorem, the required potential for $r > a$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} - \frac{qa}{b}\frac{1}{|\mathbf{r}-\mathbf{b}'|}\right\} \tag{2.2.11}$$

with $\mathbf{b}' = \mathbf{b}\,a^2/b^2$.

Induced charge density

In Chapter 1, we showed that the induced charge density on the surface of a conductor is

$$\sigma = \epsilon_0 \mathbf{E}\cdot\mathbf{n} \tag{2.2.12}$$

where $\mathbf{n}$ is the outward normal to the surface.
From Eq.(2.2.11), we have

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{q\,\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3} - \frac{qa}{b}\frac{\mathbf{r}-\mathbf{b}\,a^2/b^2}{|\mathbf{r}-\mathbf{b}\,a^2/b^2|^3}\right\} \tag{2.2.13}$$

At the surface, $|\mathbf{r}-\mathbf{b}\,a^2/b^2| = (a/b)|\mathbf{r}-\mathbf{b}|$, yielding

$$\mathbf{E}(\mathbf{r})|_{r=a} = \frac{1}{4\pi\epsilon_0}\left\{q\,\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3} - q\,\frac{\mathbf{r}-\mathbf{b}\,a^2/b^2}{(a/b)^2|\mathbf{r}-\mathbf{b}|^3}\right\} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3} - \frac{\mathbf{r}\,b^2/a^2-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right\}\,. \tag{2.2.14}$$

We see that terms proportional to $\mathbf{b}$ cancel, and the remaining terms are proportional to $\mathbf{r}$:

$$\mathbf{E}(\mathbf{r})|_{r=a} = \frac{q}{4\pi\epsilon_0}\,\mathbf{r}\,\frac{1-b^2/a^2}{|\mathbf{r}-\mathbf{b}|^3} \ . \tag{2.2.15}$$

Thus, on the surface of the $r=a$ sphere, electric field has only radial component, i.e. it is *normal* to the sphere's surface. In our case, $\mathbf{r} = a\,\hat{\mathbf{e}}_r$ and $\mathbf{n} = \hat{\mathbf{e}}_r$, thus

$$\sigma = \epsilon_0 \mathbf{E}\cdot\mathbf{n} = -\frac{q}{4\pi}\frac{a}{|a\,\hat{\mathbf{e}}_r - \mathbf{b}|^3}\left\{b^2/a^2 - 1\right\} = -\frac{q}{4\pi a}\frac{b^2-a^2}{(a^2-2ab\cos\theta+b^2)^{3/2}} \ . \tag{2.2.16}$$

Note that the surface charge density is not uniform. Still, one can verify that

$$\int_S \sigma\, dS = a^2\int_0^{2\pi} d\phi \int_0^{\pi}\sigma\,\sin\theta\, d\theta = 2\pi a^2\int_0^{\pi}\sigma\,\sin\theta\, d\theta = -qa/b = q' \ , \tag{2.2.17}$$

as expected. Indeed,

$$\int_S \sigma\, dS = -\frac{q}{4\pi}\frac{b^2-a^2}{a}\,2\pi a^2\int_0^{\pi}\frac{\sin\theta\, d\theta}{(a^2-2ab\cos\theta+b^2)^{3/2}} \ . \tag{2.2.18}$$

Changing integration variable: $\cos\theta = -z$, $\sin\theta d\theta = dz$, we get

$$\begin{aligned}\int_S \sigma\, dS &= -\frac{qa}{2}(b^2-a^2)\int_{-1}^{1}\frac{dz}{(a^2+2abz+b^2)^{3/2}} \\ &= -\frac{qa}{2}(b^2-a^2)\frac{1}{2ab}\cdot(-2)\cdot\left(\frac{1}{b+a}-\frac{1}{b-a}\right) = -q\,\frac{a}{b} = q' \ .\end{aligned} \tag{2.2.19}$$

2.2.2 Point charge near insulated conducting sphere at potential V

This is a simple modification of the method above. To increase the potential at all points on the sphere by the same amount V, we introduce an additional image charge $\hat{q} = 4\pi\epsilon_0 a\,V$ at the center of the sphere yielding $\Phi = V$ at $r=a$. Because we have introduced no additional charges in the region $r \geq a$, we apply the uniqueness theorem to say that the resultant potential is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} - \frac{qa}{b}\frac{1}{|\mathbf{r}-\mathbf{b}\,a^2/b^2|}\right\} + \frac{a}{r}\,V \ . \tag{2.2.20}$$

The total charge on the conducting sphere is now

$$Q = q' + \hat{q} = q' + 4\pi\epsilon_0 a\,V \ . \tag{2.2.21}$$

Thus the potential V on the sphere is related to the total charge on the sphere Q by

$$V = \frac{Q-q'}{4\pi\epsilon_0 a} \ . \tag{2.2.22}$$

2.2.3 Point charge near insulated, conducting sphere with total charge Q

Using Eq. (2.2.22) to express V in Eq. (2.2.20) in terms of Q, we obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} + \frac{q'}{|\mathbf{r}-\mathbf{b}\,a^2/b^2|}\right\} + \frac{Q-q'}{4\pi\epsilon_0 r} \tag{2.2.23}$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\mathbf{r}-\mathbf{b}|} + \frac{q'}{|\mathbf{r}-\mathbf{b}\,a^2/b^2|} + \frac{Q-q'}{r}\right\} . \tag{2.2.24}$$

The first term here is the potential due to the original charge q, the second term corresponds to the image charge q' located at $\mathbf{b}'$, and the third term corresponds to the image charge $Q-q'$ located at the center of the sphere.
We can now calculate the **Force** on the charge q; this is just given by Coulomb's law for the forces between q and the two image charges:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} q\left\{\frac{Q-q'}{b^3}\mathbf{b} + \frac{q'}{|\mathbf{b}-\mathbf{b}'|^3}(\mathbf{b}-\mathbf{b}')\right\} . \tag{2.2.25}$$

Using $\mathbf{b}' = \mathbf{b}\,a^2/b^2$ gives

$$\begin{aligned}\mathbf{F} =& \frac{1}{4\pi\epsilon_0}\frac{q\,\mathbf{b}}{b^3}\left\{Q - q' + \frac{q'}{|1-a^2/b^2|^3}(1-a^2/b^2)\right\} \\ =& \frac{1}{4\pi\epsilon_0}\frac{q\,\mathbf{b}}{b^3}\left\{Q - q'\left[1 - \frac{1}{(1-a^2/b^2)^2}\right]\right\} = \frac{1}{4\pi\epsilon_0}\frac{q\,\mathbf{b}}{b^3}\left\{Q + q\frac{a}{b}\left[\frac{a^4/b^4 - 2a^2/b^2}{(1-a^2/b^2)^2}\right]\right\} \\ =& \frac{1}{4\pi\epsilon_0}\frac{q\,\mathbf{b}}{b^3}\left\{Q - \frac{qa^3(2b^2-a^2)}{b(b^2-a^2)^2}\right\} .\end{aligned} \tag{2.2.26}$$

Note that, due to the induced surface charge density on the conductor, the force is **always attractive** when the distance $b-a$ is sufficiently small irrespective of Q. Namely,

$$\mathbf{F}|_{b\to a} \to -\frac{1}{4\pi\epsilon_0}\frac{q\,\mathbf{b}}{b}\frac{q}{4(b-a)^2} = -\frac{q}{4\pi\epsilon_0}\hat{\mathbf{e}}_b\frac{q}{[2(b-a)]^2} , \tag{2.2.27}$$

i.e. the force tends to that between two charges q and $-q$ separated by the distance $2(b-a)$.

2.3 Formal solution of Poisson's Equation: Preliminaries

We will now proceed to a formal solution using *Green functions*. First, however, a mathematical digression...

2.3.1 Dirac δ-Function

The *Dirac δ-function* is defined as follows:

1.

$$\delta(x-a) = 0 \qquad \text{if} \qquad x \neq a. \tag{2.3.1}$$

2.

$$\int_R dx\, \delta(x-a) = \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{otherwise} \end{cases} \tag{2.3.2}$$

The delta function is not strictly a function but rather a *distribution*; it is defined purely through its effect under an integral. It immediately follows from the definition that

$$\int dx\, f(x)\delta(x-a) = f(a) \tag{2.3.3}$$

if a lies within the region of integration.
The δ-function $\delta(x-a)$ may be thought of as the limit of a *Gaussian* centered at a in which the width tends to zero whilst the area under the Gaussian remains unity.

$$\delta(x-a) = \lim_{\epsilon\to 0} \delta_\epsilon(x-a) \tag{2.3.4}$$

$$\delta_\epsilon(x-a) = \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{(x-a)^2}{\epsilon}} \tag{2.3.5}$$

It is easy to see that $\lim_{\epsilon\to 0}\delta_\epsilon(x) = 0$ if $x \neq a$ and $\int_{-\infty}^{\infty} \delta_\epsilon(x-a) = 1$. To this end, we use the basic Gaussian integration formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{2.3.6}$$

and its modification

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\ . \tag{2.3.7}$$

A derivation of the Gaussian integration formula as simple as "$2\times 2 = 4$" proceeds as follows. Denote

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2} dx \ . \tag{2.3.8}$$

Then

$$I \times I = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-(x^2+y^2)} \ . \tag{2.3.9}$$

Treat x and y as coordinates on 2-dimensional plane, and introduce polar coordinates (ρ, φ). Then $x^2 + y^2 = \rho^2$ and $dx\, dy = \rho d\rho\, d\varphi$, which gives

$$I \times I = \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^{\infty} \underbrace{\rho\, d\rho}_{d(\rho^2)/2} \, e^{-\rho^2} = \pi \underbrace{\int_0^{\infty} d(\rho^2) e^{-\rho^2}}_{1} = \pi \ . \tag{2.3.10}$$

Since $I \times I = \pi$, we have $I = \sqrt{\pi}$.

Let us check the property (2.3.3)

$$\begin{aligned}
& \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \delta_\epsilon(x-a)\ f(x)\, dx \\
&= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{(x-a)^2}{\epsilon}} \left[f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \ldots \right] dx \\
&= \lim_{\epsilon\to 0} \left[f(a) + 0 + \frac{1}{4}\epsilon f''(a) + O(\epsilon^2) \right] dx \ = \ f(a)
\end{aligned} \tag{2.3.11}$$

Here, we noticed that the integral with $(x - a)$ has the integrand that is odd with respect to the point $x = a$, and also used the integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, x^2\, dx = -\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \ . \tag{2.3.12}$$

There are some simple relations that follow from Eq. (2.3.3)

1. The δ-function is a derivative of a *step* function $\theta(x)$:

$$\theta(x) \ = \ \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{2.3.13}$$

   Indeed, take the integral over a segment that includes $x = 0$, e.g., $-B \leq x \leq A$, with $A, B > 0$,

$$\begin{aligned}
\int_{-B}^{A} f(x)\ \theta'(x)\ dx =& f(x)\,\theta(x)\Big|_{-B}^{A} - \int_{-B}^{A} dx\, f'(x)\ \theta(x) \\
&= f(A) - \int_0^A dx\, f'(x) \\
&= f(A) - [f(A) - f(0)] = f(0)
\end{aligned} \tag{2.3.14}$$

On the other hand,

$$\int_{-B}^{A} dx\, f(x)\ \delta(x) = f(0)\ . \tag{2.3.15}$$

Hence, $\delta(x) = \theta'(x)$ (and $\delta(x-a) = \theta'(x-a)$).

2. Consider now the derivative of the delta function:

$$\begin{aligned} \int dx\, f(x)\delta'(x-a)\Big|_{\text{integ. by parts}} &= -\int dx\, f'(x)\delta(x-a) \\ &= -f'(a) \end{aligned}$$

3. Let us see what happens if we rescale the argument of the delta function: $\delta(x) \to \delta(\gamma x)$. If $\gamma > 0$, then

$$\int_{-\infty}^{\infty} f(x)\ \delta(\gamma x)\ dx = \int_{-\infty}^{\infty} f(y/\gamma)\ \delta(y)\ \frac{dy}{\gamma} = \frac{f(0)}{\gamma}\ . \tag{2.3.16}$$

So, in this case $\delta(\gamma x) = \delta(x)/\gamma$.
If $\gamma < 0$, then

$$\int_{-\infty}^{\infty} f(x)\ \delta(\gamma x)\ dx = \int_{\infty}^{-\infty} f(y/\gamma)\ \delta(y)\ \frac{dy}{\gamma} = -\frac{f(0)}{\gamma}\ . \tag{2.3.17}$$

So, in this case $\delta(\gamma x) = -\delta(x)/\gamma$.
Combining the two results, we conclude that $\delta(\gamma x) = \delta(x)/|\gamma|$.

4. Consider now the delta function with a function $g(x)$ as its argument. Obviously, only the regions of x, where $g(x)$ vanishes, are important. Let x_i's be the zeros of $g(x)$, and take $x = x_i + y$ in the vicinity of these roots. Then

$$\begin{aligned} \int dx\, f(x)\delta(g(x)) &= \sum_i \int_{-\epsilon}^{\epsilon} dy\, f(x_i+y)\delta(g(x_i+y)) \\ &= \sum_i \int_{-\epsilon}^{\epsilon} dy\, [f(x_i) + yf'(x_i) + \ldots]\delta[g(x_i) + yg'(x_i) + \ldots] \end{aligned} \tag{2.3.18}$$

Since $g(x_i) = 0$, we deal with $\delta[yg'(x_i)] = \delta(y)/|g'(x_i)|$, which gives

$$\int dx\, f(x)\delta(g(x)) = \sum_i \frac{f(x_i)}{|g'(x_i)|}\ . \tag{2.3.19}$$

5. The definition extends naturally to three (or higher) dimensions:

$$\delta(\mathbf{x} - \mathbf{X}) = \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3) \tag{2.3.20}$$

so that

$$\int_V d^3x\, \delta(\mathbf{x} - \mathbf{X}) = \begin{cases} 1 & \text{if } \mathbf{X} \in V \\ 0 & \text{otherwise} \end{cases} \tag{2.3.21}$$

Note that it is this last property that defines the multi-dimensional δ-function, with this simple representation in a Cartesian basis; you have to be a little careful when working in curvilinear coordinates.

Example. Let us show how one can use the properties of the δ-function to change integration from, say, Cartesian 2-dimension variables x, y to polar coordinates ρ, φ. To this end, we utilize that

$$\int_0^\infty d(\rho^2)\, \delta(\rho^2 - x^2 - y^2) = 1 \tag{2.3.22}$$

and

$$\int_0^{2\pi} d\varphi\, \delta\left(\sin\varphi - y/\rho\right) = \frac{2}{|\cos\varphi_0|}\ , \tag{2.3.23}$$

where φ_0 is a root of the equation $\sin\varphi = y/\rho$, and $\cos\varphi_0$ is the derivative of $(\sin\varphi - y/\rho)$ at the root. Note, that $\sin\varphi = y/\rho$ has two roots, both having the same absolute value of the derivative at the root (that explains the factor of 2 on the right hand side). Evidently, $|\cos\varphi_0| = |x|/\rho$. Thus, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ldots \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^\infty d(\rho^2)\, \delta(\rho^2 - x^2 - y^2) \int_0^{2\pi} d\varphi\, \delta\left(\sin\varphi - y/\rho\right) \frac{|x|}{2\rho} \ldots \end{aligned} \tag{2.3.24}$$

Now, it is easy to check that

$$\int_{-\infty}^{\infty} dy\, \delta\left(\sin\varphi - y/\rho\right) = \rho \tag{2.3.25}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, |x|\, \delta(\rho^2 - x^2 - y^2) &= \int_{-\infty}^{0} dx\, (-x)\, \delta(\rho^2 - x^2 - y^2) + \int_0^\infty dx\, x\, \delta(\rho^2 - x^2 - y^2) \\ &= \int_0^\infty d(x^2)\, \delta(\rho^2 - x^2 - y^2) = 1\ . \end{aligned} \tag{2.3.26}$$

Hence,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ldots = \frac{1}{2}\int_0^{\infty} d(\rho^2) \int_0^{2\pi} d\varphi \ldots$$
$$= \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi \ldots \; . \tag{2.3.27}$$

As another simple illustration of the power of the δ-function, let us return to the expression, Eq. (1.5), for the potential due to a continuous charge distribution

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{2.3.28}$$

We now introduce the δ-function to enable us to write a set of N discrete charges q_i at $\mathbf{x}_i$ as a charge distribution

$$\rho(\mathbf{x}') = \sum_i q_i \delta^{(3)}(\mathbf{x}' - \mathbf{x}_i) \tag{2.3.29}$$

so that

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\sum_i q_i \delta^{(3)}(\mathbf{x}'-\mathbf{x}_i)}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\mathbf{x}-\mathbf{x}_i|} \end{aligned} \tag{2.3.30}$$

which is our familiar expression for the potential due to a set of point charges.

Poisson's Equation for a Point Charge

We have already seen (see Eq. (2.1.4)) that

$$\nabla^2(1/r) = 0 \qquad r \neq 0. \tag{2.3.31}$$

Furthermore, from our proof of Gauss' law, we can see that

$$\int dV\, \nabla^2(1/r) = -4\pi. \tag{2.3.32}$$

Indeed, from the divergence theorem,

$$\int dV\, \nabla \cdot \nabla(1/r) = \oint_S dS\, \hat{\mathbf{n}} \cdot \nabla(1/r) \; . \tag{2.3.33}$$

Using $\nabla(1/r) = -\hat{\mathbf{e}}_r/r^2$ and choosing a sphere of radius r as the surface S (in which case $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$ and $dS = r^2\, d\Omega$), we get

$$\int dV\, \nabla^2(1/r) = -\int d\Omega = -4\pi \; . \tag{2.3.34}$$

Thus we can identify $\nabla^2(1/r) = -4\pi\delta^{(3)}(\mathbf{x})$ and write

$$\boxed{\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}') \qquad (2.3.35)}$$

2.4 Formal Solution of Boundary-Value Problem using Green Functions

Our starting point is Green's theorem, Eq. (1.11):

$$\begin{aligned}
&\int_V d^3x'[\psi_1(\mathbf{x}')\nabla'^2\psi_2(\mathbf{x}') - \psi_2(\mathbf{x}')\nabla'^2\psi_1(\mathbf{x}')] \\
&= \int_S [\psi_1(\mathbf{x}')\nabla'\psi_2(\mathbf{x}') - \psi_2(\mathbf{x}')\nabla'\psi_1(\mathbf{x}')]\cdot\mathbf{n}\, dS'. \qquad (2.4.1)
\end{aligned}$$

where the "primed" denotes differentiation with respect to the primed indices. Let us apply this for the case when $\psi_1(\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$, and thus satisfies

$$\nabla'^2\psi_1(\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')\ , \qquad (2.4.2)$$

i.e. $\psi_1(\mathbf{x}')$ is proportional to the potential produced by a point charge located at $\mathbf{x}$, while $\psi_2(\mathbf{x}') \equiv \Phi(\mathbf{x}')$ satisfies

$$\nabla'^2\Phi(\mathbf{x}') = -\rho(\mathbf{x}')/\epsilon_0\ , \qquad (2.4.3)$$

and hence may be interpreted as a potential generated by a charge distribution with density $\rho(\mathbf{x}')$. This yields

$$\begin{aligned}
&\int d^3x' \left\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{-\rho(\mathbf{x}')}{\epsilon_0}\right) + \Phi(\mathbf{x}')4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')\right\} \\
&= \int dS'\,\mathbf{n}\cdot\left\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\nabla'\Phi(\mathbf{x}') - \Phi(\mathbf{x}')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right\}. \qquad (2.4.4)
\end{aligned}$$

Applying our rule for integrating over δ-functions, we obtain a relation

$$\begin{aligned}
\Phi(\mathbf{x}) =& \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
&+ \frac{1}{4\pi}\int_S dS' \left\{\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial}{\partial n'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right\}. \qquad (2.4.5)
\end{aligned}$$

between the potential $\Phi(\mathbf{x})$ in a volume V and the values of $\Phi(\mathbf{x}')$ and $\partial\Phi(\mathbf{x}')/\partial n'$ on a surface S, which is the boundary of V. The function $1/|\mathbf{x}-\mathbf{x}'|$ is said to be a **Green function** for the problem.

The crucial property of $\psi_1(\mathbf{x}')$ that allowed us to extract $\Phi(\mathbf{x})$ (i.e. $\psi_2(\mathbf{x})$) as a separate term, was $\nabla'^2\psi_1(\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')$. The solution of this equation, and hence, the Green function is not unique: it is just a function satisfying

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{2.4.6}$$

In general, one may write it in the form

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}'), \tag{2.4.7}$$

where $F(\mathbf{x},\mathbf{x}')$ is a solution of Laplace's equation

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0. \tag{2.4.8}$$

Thus our expression for the potential can be generalized to

$$\boxed{\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') \\ &\quad + \frac{1}{4\pi}\int_{S=\partial V} dS' \left\{ G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} \right\} \quad (2.4.9)\end{aligned}}$$

The utility of this generalization is the following. In Eq.(2.4.5), the surface integral involves both $\Phi(\mathbf{x}')$, and $\partial\Phi(\mathbf{x}')/\partial n'$; in general we cannot specify *both* simultaneously at a point on the surface, since the problem is then overdetermined. Thus in Eq.(2.4.5) we have an implicit equation for $\Phi(\mathbf{x})$, with the unknown also appearing under the integral on the right-hand side. In Eq.(2.4.9), we can choose $G(\mathbf{x},\mathbf{x}')$ so that the surface integral depends only on the prescribed boundary values of Φ (Dirichlet) or $\partial\Phi/\partial n'$ (Neumann).

2.4.1 Boundary Conditions on Green Functions

We will now consider the boundary conditions we have to impose on the Green Functions to accomplish the above aim.

Dirichlet Problem

Here the value of $\Phi(\mathbf{x}')$ is specified on the surface, and therefore it is natural to impose that the Green function $G_D(\mathbf{x}, \mathbf{x}')$ satisfy

$$G_D(\mathbf{x}, \mathbf{x}') = 0 \quad \text{for } \mathbf{x}' \text{ on } S, \tag{2.4.10}$$

and thus

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x'\, G_D(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi} \int_S dS'\, \Phi(\mathbf{x}') \frac{\partial G_D(\mathbf{x}, \mathbf{x}')}{\partial n'}. \tag{2.4.11}$$

Thus the surface integral only involves $\Phi(\mathbf{x}')$, and not the unknown $\partial\Phi(\mathbf{x}')/\partial n'$.

Neumann Problem

Here it is tempting to construct the Green function $G_N(\mathbf{x}, \mathbf{x}')$ such that

$$\frac{\partial G_N(\mathbf{x}, \mathbf{x}')}{\partial n'} = 0 \quad \text{for } \mathbf{x}' \text{ on } S. \tag{2.4.12}$$

However, recall that the Green function satisfies

$$\int_S dS' \frac{\partial G_N(\mathbf{x}, \mathbf{x}')}{\partial n'} = \int d^3x'\, \nabla'^2 G_N(\mathbf{x}, \mathbf{x}') = -4\pi, \tag{2.4.13}$$

and thus $\partial G_N(\mathbf{x}, \mathbf{x}')/\partial n'$ cannot vanish everywhere. The simplest solution is to impose

$$\frac{\partial G_N(\mathbf{x}, \mathbf{x}')}{\partial n'} = -\frac{4\pi}{S}\ , \forall \mathbf{x}' \in S \tag{2.4.14}$$

where S is the total area of the surface. Thus the solution is

$$\begin{aligned} \Phi(\mathbf{x}) = {} & \frac{1}{4\pi\epsilon_0} \int_V d^3x'\, G_N(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') \\ & + \frac{1}{4\pi} \int_S dS'\, G_N(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi(\mathbf{x}')}{\partial n'} + \frac{1}{S} \int_S dS'\, \Phi(\mathbf{x}')\ , \end{aligned} \tag{2.4.15}$$

where the final term is just the average value of $\Phi(\mathbf{x}')$ on the surface S. The inclusion of this term is perhaps not surprising; recall that the solution to the Neumann problem is unique only up to an additive constant.

2.4.2 Reciprocity relation for $G_D(\mathbf{x}, \mathbf{y})$

For the Dirichlet problem, we have $G_D(\mathbf{x}, \mathbf{y}) = G_D(\mathbf{y}, \mathbf{x})$.

Proof

Apply Green's theorem for the case $\psi_1(\mathbf{x}') = G_D(\mathbf{x}, \mathbf{x}')$, and $\psi_2(\mathbf{x}') = G_D(\mathbf{y}, \mathbf{x}')$:

$$\int_V d^3x' \left(G_D(\mathbf{x}, \mathbf{x}')\nabla'^2 G_D(\mathbf{y}, \mathbf{x}') - G_D(\mathbf{y}, \mathbf{x}')\nabla'^2 G_D(\mathbf{x}, \mathbf{x}')\right) =$$
$$\int_S dS'\, \mathbf{n} \cdot \left(G_D(\mathbf{x}, \mathbf{x}')\nabla' G_D(\mathbf{y}, \mathbf{x}') - G_D(\mathbf{y}, \mathbf{x}')\nabla' G_D(\mathbf{x}, \mathbf{x}')\right).$$

But for the Dirichlet problem $G_D(\mathbf{x}, \mathbf{x}')$ vanishes for all $\mathbf{x}' \in S$, and hence the right-hand side of the above is zero. Thus we have

$$\int d^3x' \ \left\{G_D(\mathbf{x}, \mathbf{x}')\{-4\pi\delta^{(3)}(\mathbf{y} - \mathbf{x}')\} - G_D(\mathbf{y}, \mathbf{x}')\{-4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}')\}\right\} = 0 \qquad (2.4.16)$$

and hence

$$\boxed{G_D(\mathbf{x}, \mathbf{y}) = G_D(\mathbf{y}, \mathbf{x})}$$

2.5 Methods of Finding Green Functions

The secret, then, to the solution of boundary value problems is determining the correct Green function, or equivalently obtaining the function $F(\mathbf{x}, \mathbf{x}')$. There are several techniques:

1. Make a guess about the form of $F(\mathbf{x}, \mathbf{x}')$. Here we recall that F is just the solution of the homogeneous Laplace's equation $\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0$ inside V, and therefore is just the solution of the potential for a system of charges *external* to V. In particular, for the Dirichlet problem, since $G_D(\mathbf{x}, \mathbf{x}')$ vanishes at $\mathbf{x}' \in S$, we have that $F(\mathbf{x}, \mathbf{x}')$ is just the potential of that system of charges external to V which, when combined with a point charge at $\mathbf{x}$, assures that the potential vanishes on the surface. And finding that system of charges is precisely what we were doing in the *Method of Images*...

2. Expand the Green function as a series of orthonormal *eigenfunctions* of the Laplacian operator. We will be exploring this method later in the Chapter.

2.5.1 Dirichlet Green Function for the Plane

Potential for Dirichlet boundary conditions

$$\Phi(\mathbf{x}) = \int_V d^3x'\, G_D(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') - \epsilon_0 \int_S dS'\, \Phi(\mathbf{x}') \frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}. \tag{2.5.1}$$

When there are no explicit charges, then

$$\Phi(\mathbf{x}) = -\epsilon_0 \int_S dS'\, \Phi(\mathbf{x}') \frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}. \tag{2.5.2}$$

The Green function for the plane is

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}'-\mathbf{x}|} - \frac{1}{|\mathbf{x}'+\mathbf{x}|}\right], \tag{2.5.3}$$

or in components

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} - \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'+z)^2}}\right]. \tag{2.5.4}$$

Consider a boundary-value problem specified on a conducting xy plane, with a constant potential $\Phi(x,y,z=0) = V$ for $x > 0$ and $\Phi(x,y,z=0) = 0$ for $x < 0$ (the two half-planes may be considered as separated by a nonconducting line at $x = 0$).
Let us find the potential above the xy-plane, i.e. for $z > 0$. Then we have $\partial G_D(\mathbf{x},\mathbf{x}')/\partial n' = -\partial G_D(\mathbf{x},\mathbf{x}')/\partial z'$ thus we need

$$\begin{aligned}
\epsilon_0 \frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial z'} = \frac{1}{4\pi}\frac{\partial}{\partial z'}&\left[\frac{1}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{1/2}}\right. \\
&\left. - \frac{1}{[(x'-x)^2+(y'-y)^2+(z'+z)^2]^{1/2}}\right] \\
=\frac{1}{4\pi}\left(-\frac{1}{2}\right)&\left[\frac{z'-z}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}}\right. \\
&\left. - \frac{z'+z}{[(x'-x)^2+(y'-y)^2+(z'+z)^2]^{3/2}}\right].
\end{aligned} \tag{2.5.5}$$

Projecting on $z' = 0$ gives

$$\epsilon_0 \left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial z'}\right|_{z'=0} = \frac{1}{4\pi}\frac{2z}{[(x'-x)^2+(y'-y)^2+z^2]^{3/2}}. \tag{2.5.6}$$

As a result,

$$\Phi(\mathbf{x})|_{z>0} = \frac{V}{2\pi} z \int_0^\infty dx' \int_{-\infty}^\infty \frac{dy'}{[(x'-x)^2+(y'-y)^2+z^2]^{3/2}}. \tag{2.5.7}$$

Integral over y'

$$\int_{-\infty}^{\infty} \frac{dy'}{[(x'-x)^2 + (\underbrace{y'-y}_{\eta})^2 + z^2]^{3/2}} = \int_{-\infty}^{\infty} \frac{d\eta}{[\underbrace{(x'-x)^2 + z^2}_{\zeta^2} + \eta^2]^{3/2}}$$

$$= \int_{-\infty}^{\infty} \frac{d\eta}{(\zeta^2+\eta^2)^{3/2}} = 2\int_{0}^{\infty} \frac{d\eta}{(\zeta^2+\eta^2)^{3/2}} \ . \qquad (2.5.8)$$

Change variable $\eta = \zeta \tan\theta$, $d\eta = \zeta d\theta/\cos^2\theta$, and $\zeta^2+\eta^2 = \zeta^2/\cos^2\theta$. Then

$$\int_0^{\infty} \frac{d\eta}{(\zeta^2+\eta^2)^{3/2}} = \int_0^{\pi/2} \frac{\zeta\, d\theta}{\cos^2\theta}\frac{\cos^3\theta}{\zeta^3} = \frac{1}{\zeta^2}\int_0^{\pi/2} d\theta\, \cos\theta = \frac{1}{\zeta^2} = \frac{1}{(x'-x)^2+z^2} \ . \qquad (2.5.9)$$

Thus,

$$\Phi(\mathbf{x})|_{z>0} = \frac{Vz}{\pi}\int_0^{\infty} \frac{dx'}{(x'-x)^2+z^2} \ . \qquad (2.5.10)$$

Integral over x'

$$\int_0^{\infty} \frac{dx'}{(\underbrace{x'-x}_{\xi})^2+z^2} = \int_{-x}^{\infty} \frac{d\xi}{\xi^2+z^2} \ . \qquad (2.5.11)$$

Change variable $\xi = z\tan\theta$, $d\xi = z\, d\theta/\cos^2\theta$, $\xi^2+z^2 = z^2/\cos^2\theta$. Then

$$\int_{-x}^{\infty} \frac{d\xi}{\xi^2+z^2} = \int_{\tan^{-1}(-x/z)}^{\pi/2} \frac{z\, d\theta}{\cos^2\theta}\frac{\cos^2\theta}{z^2} = \frac{1}{z}\int_{-\tan^{-1}(x/z)}^{\pi/2} d\theta = \frac{1}{z}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{z}\right)\right] \ . \qquad (2.5.12)$$

Finally,

$$\Phi(\mathbf{x})|_{z>0} = \frac{V}{2} + \frac{V}{\pi}\tan^{-1}\left(\frac{x}{z}\right) \ . \qquad (2.5.13)$$

Introducing polar coordinates in xz-plane: $z = \rho\cos\phi$, $x = \rho\sin\phi \Rightarrow x/z = \tan\phi$ we have

$$\Phi(\mathbf{x})|_{z>0} = V\left[\frac{1}{2} + \frac{\phi}{\pi}\right] \ . \qquad (2.5.14)$$

Thus, equipotential surfaces correspond to planes having a constant angle ϕ with the zy-plane, with $\Phi(\phi = 0) = V/2$ on the zy-plane itself, and, as expected, $\Phi(\phi = \pi/2) = V$ on the $x > 0$ part of the xy-plane, while $\Phi(\phi = -\pi/2) = 0$ on the $x < 0$ part of the xy-plane.
Calculating electric field:

$$\mathbf{E} = -\nabla\Phi(\mathbf{x}) = -\hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi(\mathbf{x}) = -\frac{V}{\pi\rho}\,\hat{\phi} \ . \qquad (2.5.15)$$

Thus, the lines of forces are given by semi-circles going from the right half of the xy plane to its left half.

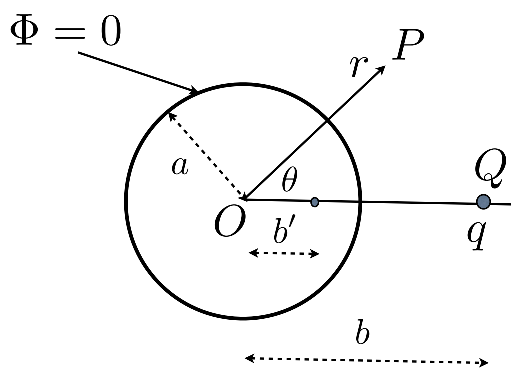
On the right half of the plane, $x > 0$, the electric field is directed vertically up, with the strength $|\mathbf{E}_{+}| = V/\pi x$, and the surface charge density there is

$$\sigma(x > 0) = \frac{|\mathbf{E}|}{\epsilon_0} = \frac{V}{\pi\epsilon_0 x} \ .$$

On the left half of the plane, $x < 0$, the electric field is directed vertically down, with the same strength $|\mathbf{E}_{-}| = V/\pi\rho = V/\pi|x|$, and the surface charge density is given there by

$$\sigma(x < 0) = -\frac{|\mathbf{E}|}{\epsilon_0} = -\frac{V}{\pi\epsilon_0|x|} = \frac{V}{\pi\epsilon_0 x} \ .$$

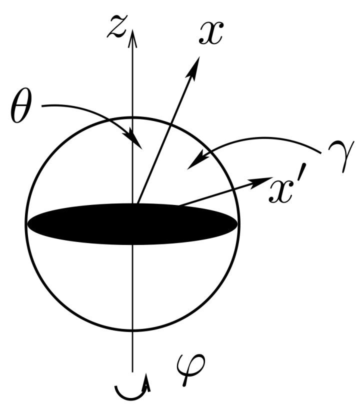
2.5.2 Dirichlet Green Function for the Sphere



We saw earlier how to use the method of images to construct the potential $\Phi(\mathbf{x}')$ for a point charge at $\mathbf{x}$ outside a grounded conducting sphere of radius a. In particular, for a charge q, at a distance $\mathbf{b}$ from the center of the sphere, the potential was given by

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{b}|} - \frac{a/b}{|\mathbf{r}-\mathbf{b}\,a^2/b^2|}\right\} \ . \tag{2.5.16}$$

Thus $\Phi(\mathbf{x}')$ is precisely the Green function $G_D(\mathbf{x}, \mathbf{x}')$ that we need. Note that you have to be careful to distinguish the variable we are integrating over, $\mathbf{x}'$, and the variable at which we are evaluating the potential, $\mathbf{x}$. Perhaps counter-intuitively, it is at the point $\mathbf{x}$ that we place our point charge.



From Eq.(2.2.11), we have that the Green function is

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{x}'-\mathbf{x}|} - \frac{a}{x|\mathbf{x}'-\mathbf{x}\,a^2/x^2|}\right\}, \tag{2.5.17}$$

and it is easy to check that, indeed, $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$.

Introducing γ as the angle between $\mathbf{x}$ and $\mathbf{x}'$, we can rewrite this as

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{(x^2x'^2/a^2+a^2-2xx'\cos\gamma)^{1/2}}\ . \tag{2.5.18}$$

Here, $\cos\gamma \equiv \mathbf{n}\cdot\mathbf{n}'$ with $\mathbf{n}$ and $\mathbf{n}'$ being unit vectors in the directions of $\mathbf{x}$ and $\mathbf{x}'$ respectively. It can be expressed in terms of the spherical polar coordinates of $\mathbf{x}$ and $\mathbf{x}'$, where

$$\mathbf{x} = x\sin\theta\cos\varphi\hat{x} + x\sin\theta\sin\varphi\hat{y} + x\cos\theta\hat{z}$$

and

$$\mathbf{x}' = x'\sin\theta'\cos\varphi'\hat{x} + x'\sin\theta'\sin\varphi'\hat{y} + x'\cos\theta'\hat{z}\ .$$

Then

$$\begin{aligned}\cos\gamma = \mathbf{n}\cdot\mathbf{n}' &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\cdot(\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta') \\ &= \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'\ . \end{aligned}\tag{2.5.19}$$

The general solution for the potential is then

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\,G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi}\int_S dS'\,\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}. \tag{2.5.20}$$

Thus we need the normal gradient of the Green function to the surface, *which points inward*,

$$\begin{aligned}\left.\frac{\partial G}{\partial n'}\right|_{\text{surface}} &= -\left.\frac{\partial G}{\partial x'}\right|_{x'=a} \\ &= \frac{1}{2}\left\{\frac{2a-2x\cos\gamma}{(x^2+a^2-2ax\cos\gamma)^{3/2}} - \frac{2x^2a/a^2-2x\cos\gamma}{(x^2a^2/a^2+a^2-2ax\cos\gamma)^{3/2}}\right\} \\ &= -\frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}\ . \end{aligned}\tag{2.5.21}$$

Thus we have all the ingredients to solve the Dirichlet problem outside a sphere of radius a.

2.5.3 Solution of Laplace's equation outside a sphere comprising two hemispheres, the upper at constant potentials V and the lower one grounded

Because the source is zero, we only need the surface term from Eq. (2.5.20)

$$\Phi(\mathbf{x}) = -\epsilon_0 \int_S dS' \, \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}. \tag{2.5.22}$$

Now $dS' = a^2 d\varphi' d(\cos\theta')$, yielding

$$\begin{aligned} \Phi(\mathbf{x}) &= -\frac{1}{4\pi} a^2 \int_0^{2\pi} d\varphi' V \int_0^1 d(\cos\theta') \frac{\partial G}{\partial n'} \\ &= \frac{V}{4\pi} \int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta') \frac{a(x^2 - a^2)}{(a^2 + x^2 - 2ax\cos\gamma)^{3/2}}. \end{aligned} \tag{2.5.23}$$

As already mentioned, we can express $\cos\gamma$ in terms of the spherical polar coordinates of $\mathbf{x}$ and $\mathbf{x}'$:

$$\cos\gamma = \sin\theta \sin\theta' \cos(\varphi - \varphi') + \cos\theta \cos\theta' \ , \tag{2.5.24}$$

giving

$$\Phi(\mathbf{x}) = \frac{V}{4\pi} a(x^2 - a^2) \int_0^{2\pi} d\varphi' \int_0^1 \frac{d(\cos\theta')}{(a^2 + x^2 - 2ax\cos\gamma)^{3/2}} \ . \tag{2.5.25}$$

In general, we cannot obtain the solution in closed form; γ is just too complicated a function of θ' and φ'. However, we can study the solution in specific cases.

Solution above North Pole

On the z-axis, $\theta = 0$, so that $\cos\gamma = \cos\theta'$, and $|\mathbf{x}| = z$. Denoting $u \equiv \cos\theta'$, we have

$$\Phi(z)|_{\theta=0} = \frac{V}{4\pi} a(z^2 - a^2) \, 2\pi \int_0^1 \frac{du}{(a^2 + z^2 - 2azu)^{3/2}} \ . \tag{2.5.26}$$

The integration can be performed easily, by making the substitution $y = a^2 + z^2 - 2azu$. Then

$$\Phi(z)|_{\theta=0} = \frac{V}{2} a(z^2 - a^2) \frac{1}{(-1/2)} \frac{1}{-2az} \left[\frac{1}{z - a} - \frac{1}{(a^2 + z^2)^{1/2}} \right] \tag{2.5.27}$$

or

$$\Phi(z)|_{\theta=0} = \frac{V}{2z} \left[z + a - \frac{z^2 - a^2}{(a^2 + z^2)^{1/2}} \right] \ , \tag{2.5.28}$$

yielding finally

$$\Phi(z)|_{\theta=0} = \frac{V}{2}\left(1+\frac{a}{z}\right)\left[1-\frac{z-a}{(a^2+z^2)^{1/2}}\right] \tag{2.5.29}$$

Note that the boundary conditions are trivially satisfied at $z = a$. Furthermore, for $z \gg a$, we have

$$\begin{aligned}\Phi(z)|_{\theta=0} =& \frac{V}{2}+\frac{Va}{2z}-V\frac{z^2-a^2}{2z(a^2+z^2)^{1/2}}\\ =& \frac{V}{2}+\frac{Va}{2z}-V\frac{z^2(1-a^2/z^2)}{2z^2(1+a^2/z^2)^{1/2}}\\ =& \frac{V}{2}+\frac{Va}{2z}-V\frac{1-a^2/z^2}{2(1+a^2/z^2)^{1/2}}\\ =& \frac{Va}{2z}+\frac{3Va^2}{4z^2}-\frac{7Va^4}{16z^4}+\dots\ .\end{aligned} \tag{2.5.30}$$

As one can see, the constant term $V/2$ disappears at large distances, and we have a $Va/2z$ behavior corresponding to the total charge

$$Q = 2\pi Va\epsilon_0$$

on the sphere.

Solution at Large Distances

We can also obtain the solution for $x \gg a$, by means of a Taylor expansion. We begin by writing

$$a^2 + x^2 \pm 2ax\cos\gamma = (a^2+x^2)(1 \pm 2\alpha\cos\gamma) \tag{2.5.31}$$

where

$$\alpha = \frac{ax}{a^2+x^2}, \tag{2.5.32}$$

yielding

$$\Phi(\mathbf{x}) = \frac{V}{4\pi}\frac{a(x^2-a^2)}{(a^2+x^2)^{3/2}}\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\left\{\frac{1}{(1-2\alpha\cos\gamma)^{3/2}}-\frac{1}{(1+2\alpha\cos\gamma)^{3/2}}\right\}. \tag{2.5.33}$$

We now expand the integrand as a power series in α, using

$$\begin{aligned}\left\{\frac{1}{(1-2\alpha\cos\gamma)^{3/2}}-\frac{1}{(1+2\alpha\cos\gamma)^{3/2}}\right\} &= 1+\left(-\frac{3}{2}\right)(-2\alpha\cos\gamma)\\ &+\frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-2\alpha\cos\gamma)^2+\frac{1}{3!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)(-2\alpha\cos\gamma)^3\\ &-\{\alpha\to-\alpha\}\ ,\end{aligned} \tag{2.5.34}$$

which yields

$$\begin{aligned}\{\} =& 1 + 3\alpha\cos\gamma + \frac{15}{2}\alpha^2\cos^2\gamma + \frac{35}{2}\alpha^3\cos^3\gamma - \{\alpha \to -\alpha\} \\ =& 6\alpha\cos\gamma + 35\alpha^3\cos^3\gamma + \mathcal{O}(\alpha^5). \end{aligned} \tag{2.5.35}$$

Note, that only odd powers of $\alpha\cos\gamma$ appear in the expansion. The integrals for the first two terms in the expansion are perfectly tractable. Recalling that

$$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta' \ ,$$

and using that the integral of $\cos(\varphi-\varphi')$ over the 2π interval vanishes, we find

$$\begin{aligned}\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos\gamma =& \int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos\theta\cos\theta' \\ =& 2\pi \times \cos\theta \times \frac{1}{2} = \pi\cos\theta \ . \end{aligned} \tag{2.5.36}$$

Similarly, using that the integral of $\cos^3(\varphi-\varphi')$ over the 2π interval also vanishes, we obtain

$$\begin{aligned}\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos^3\gamma =& \int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta') \\ &\times \left\{3\sin^2\theta\sin^2\theta'\cos^2(\varphi-\varphi')\cos\theta\cos\theta' + \cos^3\theta\cos^3\theta'\right\} \ . \end{aligned} \tag{2.5.37}$$

Since the integral of $\cos^2(\varphi-\varphi')$ over the 2π interval is equal to $(1/2)\times 2\pi = \pi$, we have

$$\begin{aligned}\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos^3\gamma =& 3\pi\sin^2\theta\cos\theta \int_0^1 d(\cos\theta')\sin^2\theta'\cos\theta' \\ &+ 2\pi\cos^3\theta \int_0^1 d(\cos\theta')\cos^3\theta' \ . \end{aligned} \tag{2.5.38}$$

Changing $\cos\theta' = \xi$, we arrive at

$$\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos^3\gamma = 3\pi\sin^2\theta\cos\theta \int_0^1 d\xi\,(1-\xi^2)\,\xi + 2\pi\cos^3\theta \int_0^1 d\xi\,\xi^3 \tag{2.5.39}$$

which gives

$$\int_0^{2\pi} d\varphi' \int_0^1 d(\cos\theta')\cos^3\gamma = \pi\cos\theta\left[\frac{3}{4}\sin^2\theta + \frac{1}{2}\cos^2\theta\right] = \frac{\pi}{4}\cos\theta(3-\cos^2\theta) \ . \tag{2.5.40}$$

Finally, combining the two results we obtain

$$\Phi(\mathbf{x}) = \frac{3Va^2x(x^2-a^2)}{2(a^2+x^2)^{5/2}}\cos\theta\left\{1 + \frac{35}{24}\frac{a^2x^2}{(a^2+x^2)^2}(3-\cos^2\theta) + \mathcal{O}(a^4/x^4)\right\}. \tag{2.5.41}$$

Note that we can express this power series as a series in a^2/x^2, rather than α, yielding

$$\Phi(x,\theta,\varphi) = \frac{3Va^2}{2x^2}\left\{\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \mathcal{O}(a^4/x^4)\right\}. \tag{2.5.42}$$

Note, that only odd powers of $\cos\theta$ appear in the final result, in accordance with the symmetry of the problem (when we change $z \to -z$, the potential Φ on the sphere changes sign). One can also verify that the expression above gives the correct expression for $\theta = 0$, i.e. on the z axis..

As we go to higher order terms in the expansion, the angular integrals become increasingly intractable, and this approach fails. However, the eagle-eyed amongst you may recognize the angular terms as the Legendre polynomials $P_1(\cos\theta)$ and $P_3(\cos\theta)$, and this brings us to the next section.

2.6 Orthogonal Functions

The expansion of the solution of a linear differential equation in terms of **orthogonal functions** is one of the most powerful techniques in mathematical physics.

Consider a set of functions $\mathcal{U}_n(\xi)$, $n = 0, 1, \ldots$, defined on $a \le \xi \le b$.

1. The set $\{\mathcal{U}_n(\xi)\}$ is **orthonormal** iff (*if and only if*)

$$\int_a^b d\xi\, \mathcal{U}_n(\xi)\mathcal{U}_m^*(\xi) = \delta_{mn}\ . \tag{2.6.1}$$

2. The set is said to be **complete** iff

$$\sum_{n=0}^{\infty} \mathcal{U}_n(\xi)\mathcal{U}_n^*(\xi') = \delta(\xi - \xi'). \tag{2.6.2}$$

The completeness relation is important because it implies that **any** square-integrable function $f(\xi)$ defined over the interval $a \le \xi \le b$ can be expressed as a series in the orthogonal functions $\mathcal{U}(\xi)$. This is easy to see:

$$\begin{aligned} f(\xi) &= \int d\xi'\, f(\xi')\delta(\xi - \xi') \quad \text{(defn. of } \delta\text{-func.)} \\ &= \int d\xi'\, f(\xi')\sum_{n=0}^{\infty}\mathcal{U}_n(\xi)\mathcal{U}_n^*(\xi') \quad \text{(completeness)} \\ &= \sum_{n=0}^{\infty}\mathcal{U}_n(\xi)\int d\xi'\, f(\xi')\mathcal{U}_n^*(\xi'). \end{aligned}$$

Thus we may write

$$f(\xi) = \sum_{n=0}^{\infty} \mathcal{U}_n(\xi) a_n \tag{2.6.3}$$

where

$$a_n = \int d\xi' f(\xi') \mathcal{U}_n^*(\xi'). \tag{2.6.4}$$

2.6.1 Fourier Series

One of the best-known cases where we expand in terms of orthogonal functions is the *Fourier expansion*. Consider the expansion applied to the interval $-a/2 \leq x \leq a/2$. The set of ortonormal functions is provided by the *sines* and *cosines*:

$$\begin{aligned}
C_m(x) &= \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, \ldots \\
S_m(x) &= \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, \ldots \\
C_0(x) &= \frac{1}{\sqrt{a}} .
\end{aligned}$$

It is easy to show that the set $C_m(x), S_m(x)$ forms an orthonormal set of functions, viz.

$$\begin{aligned}
\int dx\, S_m(x) S_n(x) = \int dx\, C_m(x) C_n(x) &= \delta_{mn}, \\
\int dx S_m(x) C_n(x) &= 0.
\end{aligned}$$

Later we will prove *completeness*,

$$\frac{1}{a} + \frac{2}{a} \sum_{m=1}^{\infty} \cos\left(\frac{2\pi m x}{a}\right) \cos\left(\frac{2\pi m x'}{a}\right) + \frac{2}{a} \sum_{m=1}^{\infty} \sin\left(\frac{2\pi m x}{a}\right) \sin\left(\frac{2\pi m x'}{a}\right) = \delta(x - x') \tag{2.6.5}$$

and thus we can write **any** function $f(x)$ on the interval $-a/2 \leq x \leq a/2$ as

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left\{ A_m \cos\left(\frac{2\pi m x'}{a}\right) + B_m \sin\left(\frac{2\pi m x'}{a}\right) \right\}, \tag{2.6.6}$$

where

$$\begin{aligned}
A_m &= \frac{2}{a} \int_{-a/2}^{a/2} dx\, f(x) \cos\left(\frac{2\pi m x}{a}\right) \quad m = 0, 1, 2, \ldots \\
B_m &= \frac{2}{a} \int_{-a/2}^{a/2} dx\, f(x) \sin\left(\frac{2\pi m x}{a}\right) \quad m = 1, 2, \ldots
\end{aligned}$$

The completeness condition may be also written as

$$\frac{1}{a} + \frac{2}{a} \sum_{m=1}^{\infty} \cos\left[\frac{2\pi m (x - x')}{a}\right] = \delta(x - x') \tag{2.6.7}$$

We can combine the sine and cosine terms by noting

$$\begin{aligned}
\cos x &= \frac{1}{2} \left[e^{ix} + e^{-ix} \right] \\
\sin x &= \frac{1}{2i} \left[e^{ix} - e^{-ix} \right],
\end{aligned}$$

and introducing a new set of functions

$$\mathcal{U}_m(x) = \frac{1}{\sqrt{a}}\, e^{i2\pi m x/a} \quad m = 0, \pm 1, \pm 2, \dots\ , \tag{2.6.8}$$

we get an expansion

$$f(x) = \sum_{m=-\infty}^{\infty} A_m\, \mathcal{U}_m(x), \tag{2.6.9}$$

where

$$A_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} dx' f(x') e^{-2\pi i m x'/a}. \tag{2.6.10}$$

Proof of completeness

$$\sum_{n=-\infty}^{\infty} e^{in(x-x')} = 2\pi\delta(x-x')$$

for $x, x' \in [-\pi, \pi]$:
For simplicity, take x instead of $x - x'$. We have

$$\sum_{n=-\infty}^{\infty} e^{inx} \;=\; \sum_{n=0}^{\infty} e^{inx} + \sum_{n=1}^{\infty} e^{-inx} \;=\; \frac{1}{1-e^{ix}} + \frac{e^{-ix}}{1-e^{-ix}} \;=\; 0$$

if $x \neq 0$. To check for the δ-function contribution, calculate

$$\int_{-\pi}^{\pi} dx \sum_{n=-\infty}^{\infty} e^{inx} \;=\; \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} dx\, e^{inx} \;=\; 2\pi$$

$\Rightarrow \sum_{-\infty}^{\infty} e^{inx} = 2\pi\delta(x)$, Q.E.D.
For the interval $[-a/2, a/2]$ we get:

$$\sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{a}(x-x')} = a\delta(x-x')\ . \tag{2.6.11}$$

Taking the real part of both sides of this equation we reproduce Eq. (2.6.7).

An orthonormal set $\sin\left(\frac{\pi}{a} m x\right)$

If we have to expand a function $f(x)$ which vanishes at the ends of the interval $[0, a]$ we can use an orthonormal set of sin's only:

$$\mathcal{U}_n(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi}{a} n x\right)\ .$$

It is easy to check that

$$\frac{2}{a}\int_0^a dx \sin\left(\frac{\pi}{a}mx\right)\sin\left(\frac{\pi}{a}nx\right) \;=\; \delta_{mn} \tag{2.6.12}$$

and

$$\frac{2}{a}\sum_{n=1}^{\infty}\sin\left(\frac{\pi}{a}nx\right)\ \sin\left(\frac{\pi}{a}nx'\right) \;=\; \delta(x-x')\ . \tag{2.6.13}$$

(Strictly speaking, in the r.h.s of Eq.(2.6.13) we get $\delta(x-x')-\delta(x+x')$ but the last term does not contribute for $x, x' \in [0,a]$).
Thus, we get an expansion

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} f_n\ \sin\left(\frac{\pi}{a}nx\right)\ , \\ f_n &= \sqrt{\frac{2}{a}}\int_0^a dx\ f(x)\sin\left(\frac{\pi}{a}nx\right)\ . \end{aligned} \tag{2.6.14}$$

2.6.2 Fourier transformation

Combining Eqs. (**??**) and (2.6.15), we have

$$f(x) = \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{2\pi imx/a}\int_{-a/2}^{a/2} dx' f(x')e^{-2\pi imx'/a}\ . \tag{2.6.15}$$

Suppose we now let $a\to\infty$, so that the discrete sum over m becomes an integral over a continuous variable k where

$$\frac{2\pi m}{a}\to k\ . \tag{2.6.16}$$

Then we have

$$\sum_m \to \frac{a}{2\pi}\int dk\ , \tag{2.6.17}$$

and Eq. (2.6.15) converts into

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx}\underbrace{\int_{-\infty}^{\infty} dx' f(x')e^{-kx'}}_{\equiv\sqrt{2\pi}A(k)}\ . \tag{2.6.18}$$

Thus the discrete coefficients become a continuous function $A(k)$ and we get the *Fourier Transforms*

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int dk\, A(k)e^{ikx}\ , \\ A(k) &= \frac{1}{\sqrt{2\pi}}\int dx'\, f(x')e^{-ikx'}. \end{aligned}$$

Note that the assignment of the coefficients outside the integrals depends on the convention adopted; in all cases the product should be equal to $1/2\pi$.

The orthogonality and completeness relations assume the continuous, and symmetric, forms

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ix(k-k')} = \delta(k-k') \tag{2.6.19}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x') \tag{2.6.20}$$

2.6.3 Sturm-Liouville Equation

How does one obtain a complete set of orthonormal functions? We will now show that, for a certain class of differential equations, the solutions are orthogonal, for specific boundary conditions.

The **Sturm-Liouville Equation** is the differential equation

$$p(x)\frac{d^2\psi_\lambda}{dx^2} + \frac{dp(x)}{dx}\frac{d\psi_\lambda}{dx} + q(x)\psi_\lambda(x) = -\lambda r(x)\psi_\lambda(x) \tag{2.6.21}$$

which we may write in the more compact form

$$\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + q(x)\psi_\lambda = -\lambda r(x)\psi_\lambda. \tag{2.6.22}$$

Here the parameter λ identifies the solution, and plays the rôle of an **eigenvalue**, with ψ_λ the corresponding **eigenvector**. In the next couple of lectures we will encounter several equations of this form – the **Legendre** and **Bessel** equations, and of course you are familiar with the time-independent **Schrödinger** equation.

2.6.4 Theorem

For the Sturm-Liouville equation, with p, q, r real functions of x, the integral

$$(\lambda^* - \lambda')\int_a^b dx\, r(x)\psi_\lambda^*(x)\psi_{\lambda'}(x) \tag{2.6.23}$$

is zero provided the following boundary condition is satisfied:

$$\left[p(x)\left(\psi_\lambda^*\frac{d\psi_{\lambda'}}{dx} - \psi_{\lambda'}\frac{d\psi_\lambda^*}{dx}\right)\right]_a^b = 0. \tag{2.6.24}$$

Proof

ψ_λ and $\psi_{\lambda'}$ satisfy

$$\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + q(x)\psi_\lambda = -\lambda r(x)\psi_\lambda \tag{2.6.25}$$

$$\frac{d}{dx}\left[p(x)\frac{d\psi_{\lambda'}}{dx}\right] + q(x)\psi_{\lambda'} = -\lambda' r(x)\psi_{\lambda'}, \tag{2.6.26}$$

respectively. Multiplying Eq. (2.6.25) by $\psi^*_{\lambda'}$ and Eq. (2.6.26) by ψ^*_λ and integrating, we obtain

$$\int_a^b \psi^*_{\lambda'}\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + \int_a^b dx\,\psi^*_{\lambda'} q\psi_\lambda = -\lambda\int_a^b dx\,\psi^*_{\lambda'} r\psi_\lambda$$

$$\int_a^b \psi^*_{\lambda}\frac{d}{dx}\left[p(x)\frac{d\psi_{\lambda'}}{dx}\right] + \int_a^b dx\,\psi^*_{\lambda} q\psi_{\lambda'} = -\lambda'\int_a^b dx\,\psi^*_{\lambda} r\psi_{\lambda'}.$$

Integrating by parts yields

$$-\int_a^b dx\,\frac{d\psi^*_{\lambda'}}{dx}p\frac{d\psi_\lambda}{dx} + \int_a^b dx\,\psi^*_{\lambda'} q\psi_\lambda = -\left[p\psi^*_{\lambda'}\frac{d\psi_\lambda}{dx}\right]_a^b - \lambda\int dx\,\psi^*_{\lambda'} r\psi_\lambda \tag{2.6.27}$$

$$-\int_a^b dx\,\frac{d\psi^*_{\lambda}}{dx}p\frac{d\psi_{\lambda'}}{dx} + \int_a^b dx\,\psi^*_{\lambda} q\psi_{\lambda'} = -\left[p\psi^*_{\lambda}\frac{d\psi_{\lambda'}}{dx}\right]_a^b - \lambda'\int dx\,\psi^*_{\lambda} r\psi_{\lambda'} \tag{2.6.28}$$

Observing that, since q, p, r are real, the l.h.s. of Eq. (2.6.27) is the complex conjugate of the l.h.s. of Eq. (2.6.28) we can take the difference to obtain

$$(\lambda^* - \lambda')\int dx\, r(x)\psi^*_\lambda\psi_{\lambda'} = 0, \tag{2.6.29}$$

providing

$$\left[p(x)\left(\psi^*_\lambda\frac{d\psi_{\lambda'}}{dx} - \psi_{\lambda'}\frac{d\psi^*_\lambda}{dx}\right)\right]_a^b = 0. \tag{2.6.30}$$

Corollaries

1. If $r(x)$ does not change sign in (a, b)

   $$\int_a^b r(x)|\psi_\lambda|^2 \neq 0 \tag{2.6.31}$$

   and hence $\lambda^* = \lambda$.

2. For $\lambda' \neq \lambda$,

   $$\int_a^b dx\, r(x)\,\psi^*_\lambda\psi_{\lambda'} = 0, \tag{2.6.32}$$

   i.e. the functions ψ_λ are **orthogonal**.

2.7 Separation of Variables in Cartesian Coordinates

We will now see how the Sturm-Liouville equation arises in the solution of Laplace's equation, and how we can then use the Sturm-Liouville theorem to provide an orthonormal set of functions. The method we will use will be the **separation of variables**. It is best shown by illustration.

Consider the solution of Laplace's equation in a box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with the values of the potential prescribed on the boundary. In particular, let us consider the case where Φ vanishes on the boundary, except on the plane $z = c$ where $\Phi(x, y, z = c) = V(x, y)$. In Cartesian coordinates, the natural coordinate system for the problem, Laplace's equation assumes the form

$$\frac{\partial^2}{\partial x^2}\Phi(x,y,z) + \frac{\partial^2}{\partial y^2}\Phi(x,y,z) + \frac{\partial^2}{\partial z^2}\Phi(x,y,z) = 0. \tag{2.7.1}$$

We will seek solutions to this equation that are **factorizable**, i.e.

$$\Phi(x,y,z) = X(x)Y(y)Z(z), \tag{2.7.2}$$

and build up our final solution from such factorizable solutions. Substituting this form into Laplace's equation, we obtain

$$\frac{d^2X(x)}{dx^2}Y(y)Z(z) + X(x)\frac{d^2Y(y)}{dy^2}X(z) + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0, \tag{2.7.3}$$

which we may write as

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{2.7.4}$$

We have separated the equation into three terms, each dependent on a different variable. Since the equation holds for all x, y, z, we can say that each term must separately be constant. Thus

$$\frac{1}{X}X'' = C_1 \tag{2.7.5}$$

$$\frac{1}{Y}Y'' = C_2 \tag{2.7.6}$$

$$\frac{1}{Z}Z'' = C_3 \tag{2.7.7}$$

where $C_1 + C_2 + C_3 = 0$.

Let us consider Eq. (2.7.5)

$$\frac{d^2X(x)}{dx^2} - C_1X = 0, \tag{2.7.8}$$

and choose a trial solution

$$X(x) = e^{\alpha x}. \tag{2.7.9}$$

Then we have that $\alpha^2 = C_1$.

1. If $C_1 > 0$, α is **real**, and the trial solution is **exponential**.

2. If $C_1 < 0$, α is **imaginary**, and the trial solution is **oscillatory**.

The boundary conditions require that X vanish at $x = 0, a$, and this is only possible for the oscillating solutions. Thus if we choose $C_1 = -\alpha^2$, where α real, the general solution will be of the form

$$X(x) = A\cos\alpha x + B\sin\alpha x. \tag{2.7.10}$$

Since X must vanish at $x = 0$,

$$X(x) = \sin\alpha x. \tag{2.7.11}$$

Furthermore, X also vanishes at $x = a$, and thus

$$\alpha = \alpha_n = \frac{n\pi}{a}, \quad n = 1, 2, \ldots. \tag{2.7.12}$$

Thus we have a set of solutions

$$X_n(x) = \sin\alpha_n x. \tag{2.7.13}$$

Eq. (2.7.5) is a Sturm-Liouville equation, with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ and $\lambda = \alpha^2$. It satisfies the conditions required for the Sturm-Liouville theorem, and hence we immediately know that the functions $X_n(x)$ are **orthogonal**. We can treat $Y(y)$ similarly, and obtain

$$Y_m(y) = \sin\beta_m y; \quad \beta_m = \frac{m\pi}{b}, m = 1, 2, \ldots \tag{2.7.14}$$

Finally, we obtain Z from

$$\frac{Z''}{Z} = \alpha_n^2 + \beta_m^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} > 0. \tag{2.7.15}$$

In this case, the solution is a real exponential, and imposing the boundary condition $Z(0) = 0$ we have

$$Z(z) = \sinh(\gamma_{nm} z) \tag{2.7.16}$$

where

$$\gamma_{nm} = \pi\sqrt{n^2/a^2 + m^2/b^2}. \tag{2.7.17}$$

Thus the general solution, using the completeness property, is

$$\Phi(x, y, z) = \sum_{m,n=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)\ . \tag{2.7.18}$$

We obtain the coefficients A_{mn} by imposing the boundary conditions on the plane $z = c$:

$$V(x, y) = \sum_{m,n=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)\ . \tag{2.7.19}$$

Using the orthonormal property of the basis functions, we have

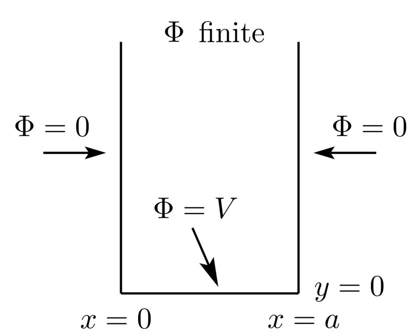
$$\begin{aligned}
&\int_0^a dx \sin\frac{n\pi x}{a}\int_0^b dy \sin\frac{m\pi y}{b} V(x, y) \\
&= \sum_{m',n'} A_{n'm'} \int_0^a dx \sin\frac{n\pi x}{a}\sin\frac{n'\pi x}{a}\int_0^b dy \sin\frac{m\pi y}{b}\sin\frac{m'\pi y}{b}\sinh\gamma_{n'm'}c \\
&= \sum_{n',m'} A_{n'm'}\frac{a}{2}\delta_{n'n}\frac{b}{2}\delta_{m'm}\sinh\gamma_{n'm'}c \\
&= \frac{ab}{4}A_{nm}\sinh\gamma_{nm}c
\end{aligned}$$

Thus we have

$$\boxed{A_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy\, V(x, y)\sin(\alpha_n x)\sin(\beta_m y)\ . \quad (2.7.20)}$$

2.7.1 Two-dimensional Square Well

This is the two-dimensional version of the above problem. We have a square well, of width a, with the potential at the bottom constrained to be $\Phi(x, 0) = V$, and zero potential on the sides, with Φ vanishing as $y \to \infty$. We wish to calculate the potential inside the well.



Laplace's equation becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{2.7.21}$$

subject to the boundary conditions

$$\begin{aligned} \Phi(0, y) = \Phi(a, y) &= 0 \\ \Phi(x, 0) &= V \\ \Phi(x, y) &\to 0 \quad \text{as } y \to \infty \end{aligned}$$

As before, we look for separable solutions $\Phi(x, y) = X(x)Y(y)$, yielding

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0, \tag{2.7.22}$$

so that each of the above terms must separately be constant.
Since $X(0) = X(a) = 0$, the solution for X must be oscillatory,

$$X'' + \alpha^2 X = 0 \tag{2.7.23}$$

giving $X(x) = \sin \alpha x$. The boundary condition at $x = a$ then yields

$$X_n(x) = \sin \alpha_n x; \text{where } \alpha_n = \tfrac{n\pi}{a}, n = 1, 2, \ldots. \tag{2.7.24}$$

The corresponding function $Y_n(y)$ must satisfy

$$Y_n'' - \alpha_n^2 Y_n = 0 \tag{2.7.25}$$

with *exponential* solutions $Y_n(y) = \exp(\pm\alpha_n y)$. The boundary condition $\Phi \to 0$ as $y \to \infty$ requires that we take the exponentially falling solution, and thus

$$Y_n(y) = e^{-\alpha_n y}. \tag{2.7.26}$$

Thus the factorizable solutions are of the form

$$\Phi_n(x, y) = e^{-\alpha_n y} \sin \alpha_n x \tag{2.7.27}$$

so that the general solution is

$$\Phi(x, y) = \sum_n A_n e^{-\alpha_n y} \sin \alpha_n x; \quad \alpha_n = \frac{n\pi}{a}. \tag{2.7.28}$$

We determine the coefficients A_n by imposing the boundary condition at $y = 0$:

$$V = \sum_n A_n \sin \alpha_n x, \tag{2.7.29}$$

and using the orthogonality of the sin functions, we obtain

$$\begin{aligned}\int_0^a V \sin \frac{n'\pi x}{a}\, dx &= \sum_n A_n \int_0^a dx\, \sin \frac{n\pi x}{a} \sin \frac{n'\pi x}{a} \\ &= \frac{a}{2} A_{n'}.\end{aligned}$$

The integral is straightforward:

$$\begin{aligned}A_n &= \frac{2V}{a} \int_0^a dx\, \sin \frac{n\pi x}{a} \\ &= -\frac{2V}{a} \frac{a}{n\pi} \left[\cos \frac{n\pi x}{a}\right]_0^a \\ &= \frac{2V}{n\pi}[1 - (-1)^n],\end{aligned}$$

and thus

$$A_n = \begin{cases} 4V/n\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{2.7.30}$$

with

$$\boxed{\Phi(x, y) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n}\, e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad (2.7.31)}$$

For $y/a \gg 1$, we can treat this as a series, and we converge to an accurate solution within a few terms - remember that exponential! To illustrate the rate of convergence, we plot the partial sum

$$\Phi_N(x, y) = \frac{4V}{\pi} \sum_{n \text{ odd } n \leq N} \frac{1}{n}\, e^{-n\pi y/a} \sin \frac{n\pi x}{a} \tag{2.7.32}$$

as a function of x for a fixed value of $y = 0.1\, a$ and for several values of N: $N = 1$ (one term), $N = 3$ (two terms), $N = 5$ (three terms) and $N = 41$ (20 terms). One can see that convergence is really fast. But in this case, we can actually sum the series.
We begin by recalling that

$$e^{ix} = \cos x + i \sin x. \tag{2.7.33}$$

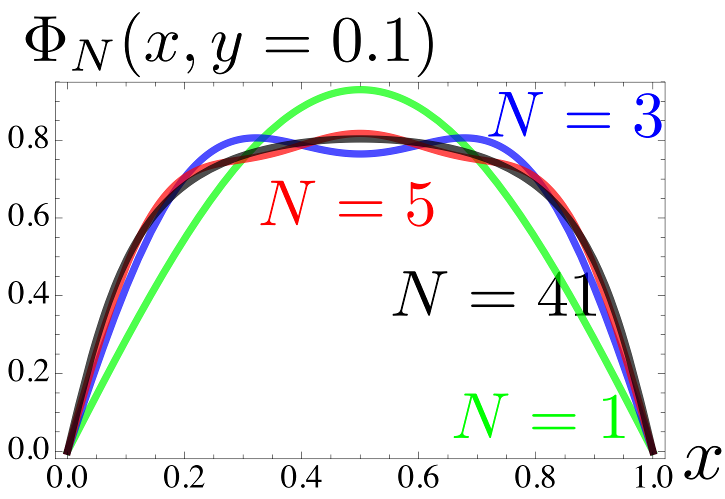


Figure 2.1: Partial sum $\Phi_N(x, y)$ given by Eq. (2.7.32) for $V = 1$, $a = 1$ and $N = 1, 3, 5, 41$ as a function of x for $y = 0.1$.

yielding

$$\sin\frac{n\pi x}{a} = \operatorname{Im} e^{in\pi x/a}. \tag{2.7.34}$$

Thus we may write the general solution as

$$\begin{aligned}\Phi(x, y) &= \frac{4V}{\pi}\sum_{n \text{ odd}}\frac{1}{n}e^{-n\pi y/a}\operatorname{Im} e^{in\pi x/a} \\ &= \frac{4V}{\pi}\sum_{n \text{ odd}}\frac{1}{n}\operatorname{Im} e^{(in\pi/a)(x+iy)} = \frac{4V}{\pi}\operatorname{Im}\sum_{n \text{ odd}}\frac{1}{n}\, e^{(in\pi/a)(x+iy)}\ .\end{aligned}$$

We now introduce the variable

$$Z = e^{(i\pi/a)(x+iy)}, \tag{2.7.35}$$

so that the solution becomes

$$\Phi = \frac{4V}{\pi}\sum_{n \text{ odd}}\frac{1}{n}\operatorname{Im} Z^n = \frac{4V}{\pi}\operatorname{Im}\sum_{n \text{ odd}}\frac{1}{n}\, Z^n\ . \tag{2.7.36}$$

To sum this series, we recall that

$$\begin{aligned}\ln(1+Z) &= Z - \frac{Z^2}{2} + \frac{Z^3}{3} + \ldots, \\ \ln(1-Z) &= -Z - \frac{Z^2}{2} - \frac{Z^3}{3} + \ldots,\end{aligned}$$

and thus

$$\sum_{n \text{ odd}} \frac{1}{n} Z^n = \frac{1}{2}\left\{\ln(1+Z) - \ln(1-Z)\right\}$$
$$= \frac{1}{2}\ln\frac{1+Z}{1-Z} \ .$$

Hence we may write the general solution as

$$\Phi(x,y) = \frac{2V}{\pi}\,\mathrm{Im}\,\ln\frac{1+Z}{1-Z}. \tag{2.7.37}$$

Now we need to write this solution explicitly in terms of x and y. We begin by denoting

$$\widetilde{Z} \equiv \frac{1+Z}{1-Z} \tag{2.7.38}$$

and writing $\widetilde{Z} = |\widetilde{Z}| \exp i\widetilde{\theta}$ where $\widetilde{\theta}$ is the phase of $\widetilde{Z}$, i.e. $\tan\widetilde{\theta} = \mathrm{Im}\,\widetilde{Z}/\,\mathrm{Re}\,\widetilde{Z}$, or

$$\widetilde{\theta} = \tan^{-1}\frac{\mathrm{Im}\,\widetilde{Z}}{\mathrm{Re}\,\widetilde{Z}} \ . \tag{2.7.39}$$

Thus

$$\ln\widetilde{Z} = \ln|\widetilde{Z}| + i\widetilde{\theta} \Longrightarrow \mathrm{Im}\,\ln\widetilde{Z} = \widetilde{\theta} = \tan^{-1}\frac{\mathrm{Im}\,\widetilde{Z}}{\mathrm{Re}\,\widetilde{Z}} \ . \tag{2.7.40}$$

Now, we need to find $\mathrm{Im}\,\widetilde{Z}$ and $\mathrm{Re}\,\widetilde{Z}$. To this end, we write

$$\frac{1+Z}{1-Z} = \frac{(1+Z)(1-Z^*)}{|1-Z|^2} = \frac{1-|Z|^2+2i\,\mathrm{Im}\,Z}{|1-Z|^2} \ , \tag{2.7.41}$$

and thus $\mathrm{Im}\,\widetilde{Z} = 2\,\mathrm{Im}\,Z/|1-Z|^2$ and $\mathrm{Re}\,\widetilde{Z} = (1-|Z|^2)/|1-Z|^2$, so that

$$\frac{\mathrm{Im}\,\widetilde{Z}}{\mathrm{Re}\,\widetilde{Z}} = \frac{2\,\mathrm{Im}\,Z}{1-|Z|^2} \tag{2.7.42}$$

This gives

$$\mathrm{Im}\,\ln\frac{1+Z}{1-Z} = \tan^{-1}\left(\frac{2\,\mathrm{Im}\,Z}{1-|Z|^2}\right). \tag{2.7.43}$$

Recall that

$$Z = e^{(i\pi/a)(x+iy)} = e^{-\pi y/a}e^{ix\pi/a} = e^{-\pi y/a}\left[\cos\frac{\pi x}{a} + i\sin\frac{\pi x}{a}\right] \ . \tag{2.7.44}$$

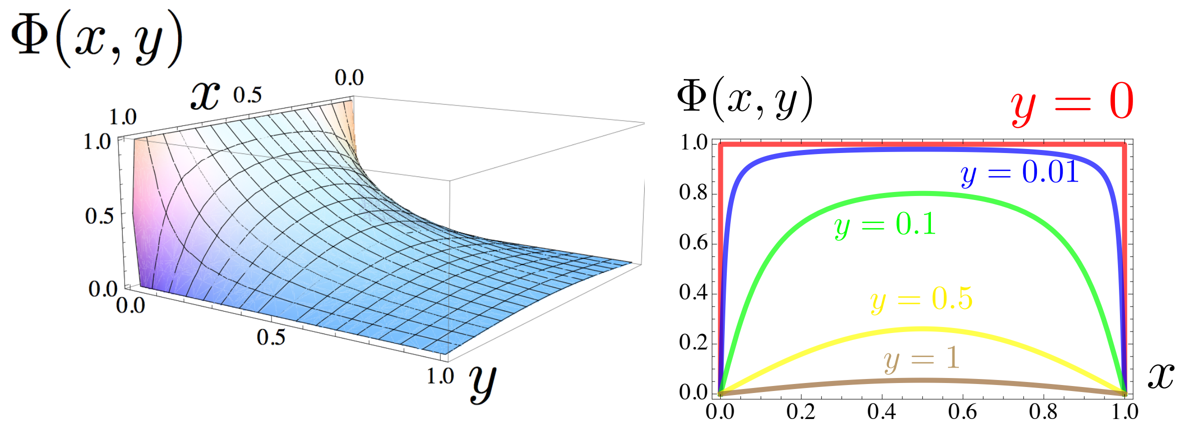


Figure 2.2: Potential $\Phi(x, y)$ given by Eq. (2.7.47) for $V = 1$ and $a = 1$ as a 3D plot and as a function of x for several values of y.

Thus, we have

$$\mathrm{Im}\, Z = e^{-\pi y/a} \sin\frac{\pi x}{a} \ , \qquad |Z|^2 = e^{-2\pi y/a} \ ,$$

and thus

$$\Phi(x, y) = \frac{2V}{\pi} \tan^{-1}\left[\frac{2e^{-\pi y/a} \sin\frac{\pi x}{a}}{1 - e^{-2\pi y/a}}\right] \ , \tag{2.7.45}$$

which, after using

$$\frac{1 - e^{-2\pi y/a}}{2e^{-\pi y/a}} = (e^{\pi y/a} - e^{-\pi y/a})/2 = \sinh(\pi y/a) \tag{2.7.46}$$

becomes

$$\boxed{\Phi(x, y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin \pi x/a}{\sinh \pi y/a}\right) . \qquad (2.7.47)}$$

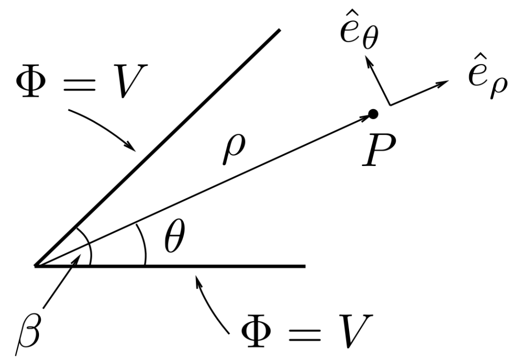
The potential $\Phi(x, y)$ for $a = 1$ and $V = 1$ is plotted in Fig. 2.7.1.
In practice, such two-dimensional problems can be done in a much simpler way, by observing that the real and imaginary components, u and v respectively, of an **analytic** complex function $f(z = x + iy)$ satisfy the two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.7.48}$$

This is a direct consequence of the Cauchy-Riemann equations.

2.7.2 Field and Charge Distribution in Two-dimensional Corners

Consider two conducting planes meeting at an angle β, with potential V on the planes. The most appropriate coordinate system for the problem is that of cylindrical polars (s, θ, z), with the z axis along the line of intersection of the planes. Note that if we consider the problem sufficiently close to the intersection, the shape of the surface at larger distances will be unimportant.



Then Laplace's equation assumes the form

$$\nabla^2\Phi(s,\theta) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\Phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{2.7.49}$$

where we have suppressed the z variable. As before we look for factorizing solutions of the form

$$\Phi(s,\theta) = R(s)T(\theta). \tag{2.7.50}$$

Then we have

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) + \frac{1}{T}\frac{\partial^2 T}{\partial\theta^2} = 0. \tag{2.7.51}$$

Each term depends on a different variable, and this must hold for all s and z. Thus each term is separably constant. For the function $T(\theta)$, let us take

$$\frac{1}{T}\frac{\partial^2 T}{\partial\theta^2} = -\nu^2. \tag{2.7.52}$$

Since T must attain the same value at $\theta = 0$ and $\theta = \beta$, the solution must be oscillatory rather than exponential, and hence ν^2 must be **positive**. Thus the solution is

$$T_\nu(\theta) = \begin{cases} A_\nu \cos\nu\theta + B_\nu \sin\nu\theta; & \nu \neq 0 \\ A_0 + B_0\theta; & \nu = 0 \end{cases} \tag{2.7.53}$$

For the radial function, we have

$$s\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) - \nu^2 R = 0. \tag{2.7.54}$$

For $\nu \neq 0$, let us take as trial solution $R \sim s^\alpha$,

$$(\alpha^2 - \nu^2)s^\alpha = 0, \tag{2.7.55}$$

yielding $\alpha = \pm\nu$. We need to consider the case $\nu = 0$ separately. Here we have

$$\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) = 0 \tag{2.7.56}$$

with solution

$$R_0(s) = a_0 + b_0 \ln s. \tag{2.7.57}$$

Thus the general form of R_ν is

$$R_\nu(s) = \begin{cases} a_\nu s^\nu + b_\nu s^{-\nu}; & \nu > 0 \\ a_0 + b_0 \ln s; & \nu = 0 \end{cases} , \tag{2.7.58}$$

and the general solution for the potential has the form

$$\Phi(s,\theta) \;=\; (a_0 + b_0 \ln s)(A_0 + B_0\theta) + \sum_{\nu>0}(a_\nu s^\nu + b_\nu s^{-\nu})(A_\nu \cos\nu\theta + B_\nu \sin\nu\theta)\ . \tag{2.7.59}$$

The solution must be valid as $s \to 0$ (note that we are not interested in the solution for s large), and therefore the terms proportional to $\ln s$ and $s^{-\nu}$ cannot contribute. Thus $b_0 = 0$ and $b_\nu = 0$. We can also take $a_0 = 1$, which amounts to redefining $a_0 A_0$ into A_0. Then our solution is of the form

$$\Phi(s,\theta) \;=\; A_0 + B_0\theta + \sum_{\nu>0} a_\nu s^\nu (A_\nu \cos\nu\theta + B_\nu \sin\nu\theta)\ . \tag{2.7.60}$$

We will now use the boundary conditions on the planes to further constrain the solution. At $\theta = 0$ we have

$$\Phi(s,0) = V = A_0 + \sum_{\nu>0} a_\nu s^\nu A_\nu\ , \tag{2.7.61}$$

i.e., $\Phi = V$, independent of s, and therefore $A_0 = V$ and $A_\nu = 0$. Now, at $\theta = \beta$ we also have $\Phi = V$, independent of s, or

$$\Phi(s,\beta) \;=\; V = A_0 + B_0\beta + \sum_{\nu>0} a_\nu s^\nu B_\nu \sin\nu\beta\ . \tag{2.7.62}$$

The s-dependence disappears only if $a_\nu B_\nu \sin\nu\beta$ (for $\nu > 0$) vanish, which (for a nontrivial solution) happens if $\sin\nu\beta = 0$, that requires

$$\nu = \frac{n\pi}{\beta},\ n = 1, 2, \dots\ .$$

Then we have $V = A_0 + B_0\beta$. Now, since we have already established that $A_0 = V$, we conclude that $B_0 = 0$. Thus our final result (after redefinition $a_\nu B_\nu \to B_n$) is given by

$$\boxed{\Phi(s,\theta) = V + \sum_{n=1}^{\infty} B_n s^{n\pi/\beta} \sin \frac{n\pi\theta}{\beta} \ . \qquad (2.7.63)}$$

As we get closer into the corner, $s \to 0$, the first term will dominate, and

$$\Phi(s,\theta) \sim V + B_1 s^{\pi/\beta} \sin \frac{\pi\theta}{\beta} \ . \tag{2.7.64}$$

Taking the gradient, we obtain

$$\begin{aligned} \mathbf{E} = -\nabla\Phi &= -\frac{\partial\Phi}{\partial s}\mathbf{e_s} - \frac{1}{s}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta \\ &= -\frac{\pi B_1}{\beta} s^{\pi/\beta - 1} \sin \frac{\pi\theta}{\beta}\mathbf{e_s} - \frac{\pi B_1}{\beta} s^{\pi/\beta-1} \cos \frac{\pi\theta}{\beta}\mathbf{e}_\theta \ . \end{aligned} \tag{2.7.65}$$

Note that for $\theta = 0$ and $\theta = \beta$, the electric field does not have radial component, i.e. $\mathbf{E}$ is normal to the surface of conductor. Also, because $\cos \frac{\pi\theta}{\beta} = 1$ for $\theta = 0$ and $\cos \frac{\pi\theta}{\beta} = -1$ for $\theta = \beta$, the field on both surfaces is oriented (for positive B_1) from the interior of the angle toward the conductor. Induced surface charge density is given by

$$\sigma = \epsilon_0 \left(\mathbf{E} \cdot \mathbf{n}\right) = -\frac{\pi B_1 \epsilon_0}{\beta} s^{\pi/\beta - 1} \ . \tag{2.7.66}$$

Now observe that

1. For $\beta < \pi$, we have that $\mathbf{E}$ and σ vanish as $s \to 0$.

2. For $\beta > \pi$, $\mathbf{E}$ and σ become **singular** as $s \to 0$.

Thus we see behaviour familiar from our knowledge of “action at points” – the fields and surface charge densities become singular near sharp edges.