604 Final Exam (34 points). 12/15/11, 15:45 - 18:45

Problem 1.

A pure electric dipole is located at a certain distance above an infinite grounded conducting plane. If the dipole is free to rotate, in what orientation it will come to rest?

Solution

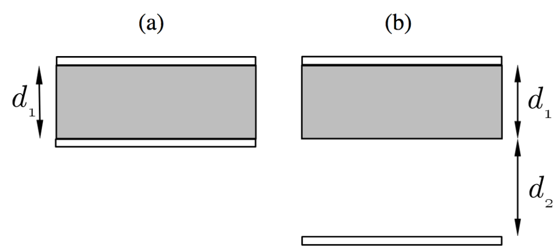
First, let us calculate the potential energy of the dipole oriented along the θ, ϕ direction in spherical polar coordinates. The mirror image of this dipole is a dipole located a distance d below the $z = 0$ surface and oriented along $\theta, \pi + \phi$ direction. The potential energy is

$$U \;=\; \frac{1}{32\pi\epsilon_0 d^3}[(\vec{p}_1 \cdot \vec{p}_2) - 3(\vec{p}_1 \cdot \vec{e}_3)(\vec{p}_2 \cdot \vec{e}_3)] \;=\; -\frac{p^2}{32\pi\epsilon_0 d^3}(1 + \cos^2\theta)$$

The dipole will come to rest in $\theta = 0$ or $\theta = \pi$ orientation depending whether the original angle θ was smaller or greater than $\pi/2$.

Problem 2

A parallel plate capacitor of plate separation d_1 is filled with a solid dielectric material of susceptibility χ_e as shown in the figure (a) below The capacitor is charged to voltage V_1. The capacitor is then disconnected from the battery and pulled apart so that the plate separation becomes $d_1 + d_2$. The dielectric does not expand and the dielectric-free region has size d_2, see figure (b). Assuming the plates are large compared to both d_1 and d_2, compute the voltage V_2 after the capacitor is pulled apart.



Solution

This is a capacitors-in-series setup with $C_1 = A\frac{\epsilon}{d_1}$ and $C_2 = A\frac{\epsilon_0}{d_2}$. Since $C_{12} = \frac{C_1 C_2}{C_1 + C_2}$ and the charge $Q = C_1 V_1$ does not change,

$$V_2 \;=\; \frac{Q}{C_{12}} \;=\; \frac{C_1 + C_2}{C_2} V_1 \;=\; \Big(1 + (1 + \chi_e)\frac{d_2}{d_1}\Big)$$

Problem 3

A surface (free) charge is glued over the surface of the dielectric sphere of radius R in such a way that the potential outside the sphere is

$$\Phi_{\text{out}}(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\frac{3\cos^2\theta - 1}{2r^3}$$

Find (surface) charge density of free charge and densities of bound charges. Susceptibility of the dielectric is χ_e.

Solution

The potential inside is

$$\Phi_{\rm in}(\vec{r}) \;=\; \sum_{l=0} A_l r^l P_l(\cos\theta)$$

Since the only input is $\sim P_2(\cos\theta)$ we get

$$\Phi_{\rm in}(\vec{r}) \;=\; A_2 r^2 P_2(\cos\theta) \;=\; \frac{A_2}{2} r^2 (3\cos^2\theta - 1)$$

The tangential component $E_\theta = -\frac{\partial \Phi}{\partial \theta}$ is continuous so

$$A_2 \;=\; \frac{1}{4\pi\epsilon_0 R^5} \qquad \Rightarrow \qquad \Phi_{\rm in}(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\frac{r^2}{R^5}\frac{3\cos^2\theta - 1}{2}$$

The surface charge density is proportional to discontinuity of D_r so

$$\sigma_f \;=\; D_r^{\rm out} - D_r^{\rm in} \;=\; -\,\epsilon_0 \,\frac{\partial \phi_{\rm out}(r,\theta)}{\partial r}\bigg|_{r\to R} + \epsilon\, \frac{\partial \phi_{\rm in}(r,\theta)}{\partial r}\bigg|_{r\to R} \;=\; \frac{1}{4\pi R^4}\Big(\frac{5}{2} + \chi_e\Big)(3\cos^2\theta - 1)$$

To find bound charges, let us first write down $\vec{E}$ inside

$$\vec{E} \;=\; \frac{r}{4\pi\epsilon_0 R^5}\big[-(3\cos^2\theta - 1)\hat{e}_r + 3\sin\theta\cos\theta\hat{e}_\theta\big]$$

Next, polarization is

$$\vec{P} \;=\; \chi_e\epsilon_0\vec{E} \;=\; \frac{\chi_e}{4\pi R^5} r\big[-(3\cos^2\theta - 1)\hat{e}_r + 3\sin\theta\cos\theta\hat{e}_\theta\big]$$

so

$$\sigma_b \;=\; \vec{P}\cdot\hat{r}\Big|_{r=R} \;=\; -\,\frac{\chi_e}{4\pi R^4}(3\cos^2\theta - 1), \qquad \rho_b \;=\; -\,\nabla\cdot\vec{P} \;=\; -\,\frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} \;=\; -\,\frac{\chi_e}{1+\chi_e}\sigma_f \;=\; 0$$

Problem 4.

Consider an infinite cylinder $x^2 + y^2 \le a^2$ and $z \ge 0$. The cylindrical surface is grounded while the potential at $z = 0$ is $V(r,\phi)$. Find the potential at $z > 0$ (assume that the potential vanishes as $z \to \infty$).

The solution for final cylinder of length L grounded at $z = L$ is

$$\begin{aligned}\Phi(s,\varphi,z) \;&=\; \sum_{n=1}^{\infty}\frac{B_{0n}}{2} J_0(k_{0n}s)\sinh\big(k_{0n}(L-z)\big)\\ &+\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s)\sinh\big(k_{mn}(L-z)\big)\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi].\end{aligned}$$

where

$$\begin{aligned}A_{mn} \;&=\; \frac{2}{\pi a^2\sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2}\int_0^a ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s)\sin m\varphi\\ B_{mn} \;&=\; \frac{2}{\pi a^2\sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2}\int_0^a ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s)\cos m\varphi\end{aligned}$$

As $L \to \infty$ we get

$$\Phi(s,\varphi,z) \;=\; \sum_{n=1}^{\infty}\frac{b_{0n}}{2} J_0(k_{0n}s) e^{-k_{0n}z} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s) e^{-k_{mn}z}\,[a_{mn}\sin m\varphi + b_{mn}\cos m\varphi].$$

where

$$a_{mn} = \frac{2}{\pi a^2 [J_{m+1}(x_{mn})]^2} \int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s) \sin m\varphi$$
$$b_{mn} = \frac{2}{\pi a^2 [J_{m+1}(x_{mn})]^2} \int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s) \cos m\varphi$$

Problem 5.

Consider the vector potential $\vec{A}(\vec{r}) = \frac{1}{2}\vec{a} \times \vec{r}$ ($\vec{a}$ is a constant vector).
a) What is the magnetic field?
b) Does this vector potential satisfy Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$? If not, modify $\vec{A}$ such that the magnetic field remains unchanged but the Coulomb condition is satisfied.

Solution

If $\vec{A} = \frac{1}{2}\vec{a} \times \vec{r}$

$$\vec{B} = \frac{1}{2}\vec{\nabla} \times (\vec{a} \times \vec{r}) = \frac{1}{2}\vec{a}(\vec{\nabla} \cdot \vec{r}) - \frac{1}{2}(\vec{a} \cdot \vec{\nabla})\vec{r} = \frac{3}{2}\vec{a} - \frac{1}{2}\vec{a} = \vec{a}$$

Coulomb gauge condition is satisfied:

$$\frac{1}{2}\vec{\nabla} \cdot (\vec{a} \times \vec{r}) = \frac{1}{2}\partial_i \epsilon_{ijk} a_j x_k = 0$$

Problem 6.

An infinitely long ferromagnetic cylinder, of radius R, carries a “frozen-in” magnetization parallel to the axis, $\vec{M} = ks\hat{e}_3$ where k is constant and s is the distance from the axis. (There is no free current anywhere). Find
a) The magnetic field $\vec{B}$ inside and outside the cylinder.
b) Surface and volume bound currents

Solution

Since there is no free current we can use cylindrical symmetry to prove that $\vec{H} = 0$ everywhere. For the Amperian circle with radius s one gets

$$\oint \vec{H} \cdot d\vec{l} = H(s) 2\pi s = (I_{\text{free}})_{\text{enc}} = 0$$

$\Rightarrow \vec{H} = 0$. Using the definition $\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$ we get $\vec{B} = 0$ outside the cylinder and $\vec{B} = \mu_0\vec{M} = k\mu_0 s\hat{e}_3$ inside.
The bound currents are

$$\vec{K}_{\text{bound}} = \vec{M} \times \vec{s} = kR\hat{\phi}$$
$$\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M} = -k\hat{\phi}$$