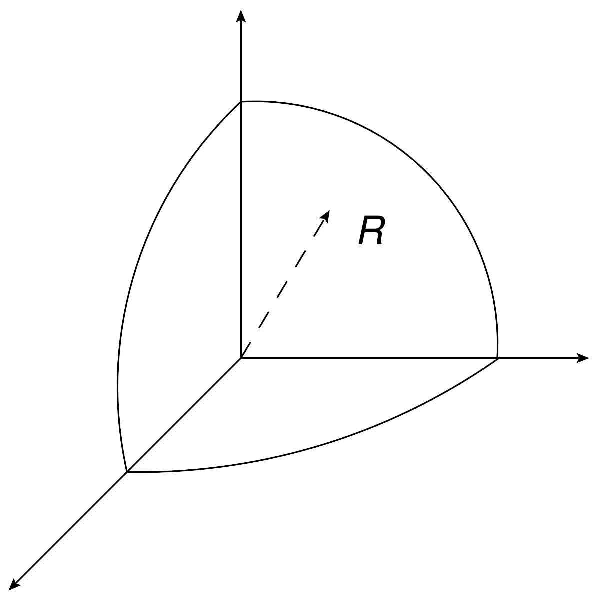
HW 3 solution

Problem 1.

Consider 1/8 of a spherical shell of radius R (that is, surface with $r = R$ and $x, y, z \geq 0$) uniformly charged with surface density σ. Find the potential and the electric field at the origin.



Solution.

The general formula for an electric potential due to surface charge distribution reads

$$\Phi(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|} \tag{1}$$

In our case $|\vec{0}-\vec{r}'| = R$ so

$$\Phi(\vec{0}) \;=\; \frac{1}{4\pi\epsilon_0 R}\int_S \sigma(\vec{r}') \;=\; \frac{Q}{4\pi\epsilon_0 R} \;=\; \frac{\sigma\pi R^2/2}{4\pi\epsilon_0 R} \;=\; \frac{\sigma R}{8\epsilon_0} \tag{2}$$

For the electric field, the general formula has the form

$$\vec{E}(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r}')(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r}'|^3} \tag{3}$$

In our case $\sigma(\vec{r}')$=const so

$$\vec{E}(\vec{0}) \;=\; \frac{\sigma}{4\pi\epsilon_0}\int_S dS \frac{\vec{0}-\vec{r'}}{|\vec{0}-\vec{r}'|^3} \;=\; -\frac{\sigma}{4\pi\epsilon_0 R^3}\int_S dS\; \vec{r'} \;=\; -\frac{\sigma}{4\pi\epsilon_0 R}\int_0^{\frac{\pi}{2}} \sin\theta\; d\theta \int_0^{\frac{\pi}{2}} d\phi\; \vec{r'} \tag{4}$$

$$= \; -\frac{\sigma}{4\pi\epsilon_0 R^3}\int_S dS\; \vec{r'} \;=\; -\frac{\sigma}{4\pi\epsilon_0}\int_0^{\frac{\pi}{2}} \sin\theta\; d\theta \int_0^{\frac{\pi}{2}} d\phi\; (\hat{e}_1 \sin\theta\cos\phi + \hat{e}_2 \sin\theta \sin\phi + \hat{e}_3\cos\theta) \;=\; \frac{\sigma}{16\epsilon_0}(\hat{e}_1+\hat{e}_2+\hat{e}_3)$$

Problem 2.

Solve the same problem for 1/8 of a sphere (ball) defined by $r \leq R$ and $x, y, z > 0$. Assume that it is uniformly charged with volume density ρ.

Solution.

The easiest way is to use our results for the spherical shell. Consider part of the body shown in Fig. 1 with radii between r and $r + dr$. It can be approximated with charged spherical shell with radius r and surface density $\sigma = \rho dr$. Due to Eq. (2) the contribution of this shell to the potential at the origin is

$$d\Phi \;=\; \frac{\rho}{8\epsilon_0} r dr$$

and therefore

$$\Phi(\vec{0}) \;=\; \int_0^R \frac{\rho}{8\epsilon_0} r dr \;=\; \frac{\rho R^2}{16\epsilon_0} \tag{5}$$

Similarly, the contribution of "shell" between r and $r + dr$ to the electric field at the origin can be taken from Eq. (4)

$$d\vec{E} \;=\; -\left(\hat{e}_1 + \hat{e}_2 + \hat{e}_3\right) \frac{\rho}{16\epsilon_0} dr$$

and therefore

$$\vec{E}(\vec{0}) \;=\; -\left(\hat{e}_1 + \hat{e}_2 + \hat{e}_3\right) \frac{\rho}{16\epsilon_0} \int_0^R dr \;=\; -\frac{\rho R}{16\epsilon_0} \left(\hat{e}_1 + \hat{e}_2 + \hat{e}_3\right) \tag{6}$$

$$\Phi(\vec{0}) \;=\; \int_0^R \frac{\rho}{8\epsilon_0} r dr \;=\; \frac{\rho R^2}{16\epsilon_0} \tag{7}$$

The potential and electric field can also be obtained by direct integration

$$\Phi(\vec{0}) \;= \frac{1}{4\pi\epsilon_0} \int_V d^3x' \frac{\rho(\vec{r'})}{|\vec{r'}|} \;=\; \frac{\rho}{4\pi\epsilon_0} \int_0^R r'^2 dr' \int_0^{\frac{\pi}{2}} \sin\theta \; d\theta \int_0^{\frac{\pi}{2}} d\phi \; \frac{1}{r'} \;=\; \frac{\rho R^2}{16\epsilon_0} \tag{8}$$

$$\begin{aligned} \vec{E}(\vec{0}) \;&=\; \frac{\rho}{4\pi\epsilon_0} \int_V d^3x' \frac{\vec{0} - \vec{r'}}{|\vec{0} - \vec{r'}|^3} \;=\; -\frac{\rho}{4\pi\epsilon_0} \int_V dV \; \frac{1}{r'^2} \left(\hat{e}_1 \sin\theta\cos\phi + \hat{e}_2 \sin\theta\sin\phi + \hat{e}_3 \cos\theta\right) \;= \\ &=\; -\frac{\rho}{4\pi\epsilon_0} \int_0^R dr' \int_0^{\frac{\pi}{2}} \sin\theta \; d\theta \int_0^{\frac{\pi}{2}} d\phi \; \left(\hat{e}_1 \sin\theta\cos\phi + \hat{e}_2 \sin\theta\sin\phi + \hat{e}_3 \cos\theta\right) \;=\; -\frac{\rho R}{16\epsilon_0} \left(\hat{e}_1 + \hat{e}_2 + \hat{e}_3\right) \end{aligned} \tag{9}$$