HW 6 solution

Problem 2.23 a
In Cartesian coordinate

$$\frac{\partial^2}{\partial x^2}\Phi(x,y,z)+\frac{\partial^2}{\partial y^2}\Phi(x,y,z)+\frac{\partial^2}{\partial z^2}\Phi(x,y,z)=0. \tag{1}$$

We will seek solutions to this equation that are **factorizable**, i.e.

$$\Phi(x,y,z)=X(x)Y(y)Z(z), \tag{2}$$

Similarly to Lecture 9, since X must vanish at $x=0$,

$$X(x)=\sin\alpha x. \tag{3}$$

Furthermore, X also vanishes at $x=a$, and thus

$$\alpha=\alpha_n=\frac{n\pi}{a},\quad n=1,2,\ldots. \tag{4}$$

Thus we have a set of solutions

$$X_n(x)=\sin\alpha_n x. \tag{5}$$

We can treat $Y(y)$ similarly, and obtain

$$Y_m(y)=\sin\beta_m y;\quad \beta_m=\frac{m\pi}{a}, m=1,2,\ldots \tag{6}$$

Finally, we obtain Z from

$$\frac{Z''}{Z}=\alpha_n^2+\beta_m^2=\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{a^2}>0. \tag{7}$$

It is convenient to consider the shifted cube $a>x,y>0$ and $\frac{a}{2}>z>-\frac{a}{2}$
In this case, the solution is symmetric under $z\leftrightarrow -z$ which singles out the solution

$$Z(z)=A_{mn}\cosh(\gamma_{nm}z) \tag{8}$$

where

$$\gamma_{nm}=\pi\sqrt{n^2/a^2+m^2/a^2}. \tag{9}$$

Thus the general solution, using the completeness property, is

$$\Phi(x,y,z)=\sum_{m,n=1}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)A_{nm}\cosh(\gamma_{nm}z) \tag{10}$$

We obtain the coefficients A_{mn} by imposing the boundary conditions on the plane $z=\pm\frac{a}{2}$

$$V=\sum_{m,n=1}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)A_{nm}\cosh(\gamma_{nm}a/2) \tag{11}$$

Using the orthonormal property of the basis functions, we have at $z = 0$

$$\begin{aligned}
&\int_0^a dx \sin\frac{n\pi x}{a}\int_0^b dy \sin\frac{m\pi y}{b}\; V \\
&= \sum_{m',n'} A_{n'm'}\int_0^a dx \sin\frac{n\pi x}{a}\sin\frac{n'\pi x}{a}\int_0^a dy \sin\frac{m\pi y}{a}\sin\frac{m'\pi y}{a}\cosh\gamma_{n'm'}\frac{a}{2} \\
&= \sum_{n',m'} A_{n'm'}\frac{a}{2}\delta_{n'n}\frac{a}{2}\delta_{m'm}\cosh\gamma_{n'm'}\frac{a}{2} \\
&= \frac{a^2}{4}A_{nm}\cosh\big(\gamma_{nm}\frac{a}{2}\big)
\end{aligned}$$

We get

$$A_{nm} = \frac{4V}{a^2\cosh(\gamma_{nm}\frac{a}{2})}\int_0^a dx\int_0^a\, dy\, \sin(\alpha_n x)\sin(\beta_m y) \;=\; \frac{16V}{\pi^2 mn\cosh\frac{\pi}{2}\sqrt{m^2+n^2}}\;. \tag{12}$$

if m and n are odd and $A_{mn} = 0$ if either m or n is even.

Thus, the cube $a > x, y > 0$ and $\frac{a}{2} > z > -\frac{a}{2}$

$$\Phi(x,y,z) \;=\; V\sum_{m,n=odd}^{\infty}\frac{16}{\pi^2 mn}\sin\frac{\pi nx}{a}\sin\frac{\pi my}{a}\;\frac{\cosh\frac{\pi}{a}\sqrt{m^2+n^2}z}{\cosh\frac{\pi}{2}\sqrt{m^2+n^2}} \tag{13}$$

After the shift $z \to z + \frac{a}{2}$

$$\Phi(x,y,z) \;=\; V\sum_{m,n=odd}^{\infty}\frac{16}{\pi^2 mn}\sin\frac{\pi nx}{a}\sin\frac{\pi my}{a}\;\frac{\cosh\frac{\pi}{2}\;\sqrt{m^2+n^2}\big(2\frac{z}{a}-1\big)}{\cosh\frac{\pi}{2}\sqrt{m^2+n^2}} \tag{14}$$

Proble 2.23 c

The charge density iat $z = a$

$$\sigma \;=\; \epsilon_0 E \;=\; \epsilon_0\frac{\partial\Phi}{\partial z} \;=\; \frac{16V\epsilon_0}{a\pi}\sum_{m,n=odd}^{\infty}\frac{\sqrt{m^2+n^2}}{mn}\sin\frac{\pi nx}{a}\sin\frac{\pi my}{a}\;\tanh\frac{\pi}{2}\sqrt{m^2+n^2} \tag{15}$$