HW assignment 9 Problem 3.9 from *Jackson*.

Solution. By separation of variables we get

$$\phi(s,\varphi,z) \;=\; R(s)T(\varphi)Z(z)$$

where $T(\varphi) \;=\; e^{\pm im\varphi}$ and

$$Z(z) \;=\; e^{\pm kz}, \qquad R(s) \;=\; J_m(ks),\; N_m(ks)$$

or

$$Z(z) \;=\; e^{\pm ikz}, \qquad R(s) \;=\; I_m(ks),\; K_m(ks)$$

depending on boundary conditions. In our case $\phi(s,\varphi,z)\big|_{z=0,L} = 0$ so we need the second set with

$$Z(z) \;=\; \sin\frac{\pi n z}{L} \quad \Leftrightarrow \quad k_n \;=\; \frac{\pi n}{L}$$

Moreover, there are no charges inside the cylinder so the potential must be regular there and therefore $K_m(ks)$ are not acceptable. Thus, our solution has the form

$$\phi(s,\varphi,z) \;=\; \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} c_{mn} I_m(k_n s)\sin k_n z e^{im\varphi}$$

To find c_{mn} we take $s = b$ and integrate both sides of this equation over angle φ

$$\int_0^{2\pi} d\varphi\; e^{iM\varphi} V(\varphi,z) \;=\; 2\pi \sum_{n=1}^{\infty} c_{Mn} I_M(\frac{\pi n b}{L})\sin\frac{\pi n}{L} z$$

and over z using $\int_0^L \sin(\frac{\pi m}{L}z)\sin(\frac{\pi n}{L}z) \;=\; \frac{L}{2}\delta_{mn}$

$$\int_0^L dz\;\sin(\frac{\pi N}{L}z)\int_0^{2\pi} d\varphi\; e^{iM\varphi}V(\varphi,z) \;=\; \pi L c_{MN} I_M(\frac{\pi N b}{L})$$

and therefore

$$c_{mn} \;=\; \frac{1}{\pi L I_m(\frac{\pi n b}{L})}\int_0^L dz\;\sin(\frac{\pi n}{L}z)\int_0^{2\pi} d\varphi\; e^{im\varphi}V(\varphi,z)$$