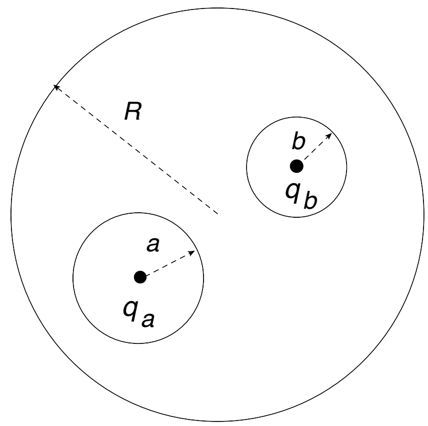
Problem 1.

Two spherical cavities of radii a and b, respectively, are hollowed out from the interior of a neutral metal sphere of radius R as shown below. Point charges q_a and q_b are placed at the centers of the cavities. Find the surface charge densities σ_a and σ_b on the surfaces of the two cavities and the surface charge density σ_R the outer surface of the metal sphere.



Solution

We use superposition principle solve three problems: charge q_a in the cavity hollowed out of an infinite conductor with potential 0, same for charge q_b, and conducting sphere of radius R with charge $q_a + q_b$. The solution of the first problem satisfies Poission equation in the first cavity and has potential $\Phi_1 = 0$ everywhere else, same for the second with $\Phi_2 = 0$ outside the second cavity, and the solution of the third problem satisfies Laplace equation outside the sphere with radius R and has constant potential $\Phi_3 = \frac{q_a+q_b}{4\pi\epsilon_0 R}$ inside that sphere.

The superposition of these three problems has the potential

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 \qquad (*)$$

which is constant (=Φ_3) throughout our conductor with two cavities. In addition, the potential inside the first cavity is

$$\Phi = \frac{q_a}{4\pi\epsilon_0|\vec{r} - \vec{r}_a|} + \Phi_3$$

which obviously satisfies Poisson equation

$$\nabla^2\Phi = -\frac{q_a}{\epsilon_0}\delta(\vec{r} - \vec{r}_a)$$

since Φ_3=const there. Similarly, the potential inside the second sphere satisfies correct Poisson equation, and the potential outside the sphere of radius R satisfies Laplace equation. Thus, our superposition $(*)$ satisfies correct Poisson equations and boundary conditions that the potential throughout the conductor is constant.

Finally, the surface charge on the first cavity is the same as in auxiliiary problem

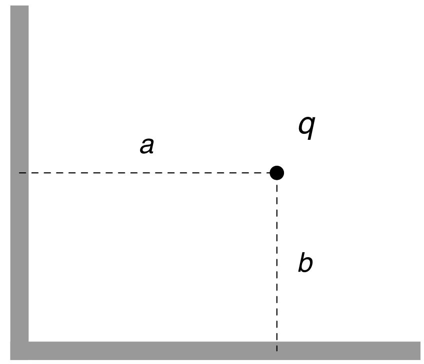
$$\sigma_a = -\frac{q_a}{4\pi a^2}$$

and similarly

$$\sigma_b = -\frac{q_b}{4\pi b^2}$$

As seen from the third auxiliary problem, the charge on the outer surface of the sphere is distributed symmetrically so

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

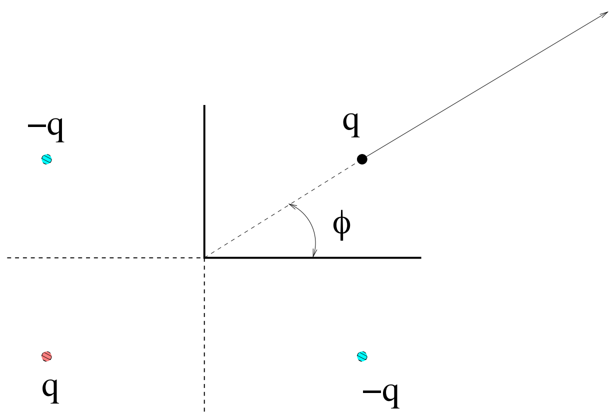


Problem 2.

Two semi-infinite grounded conducting plates meet at right angles. How much work does it take to bring a point charge q from infinity to the point located at distance a from the first plate and distance b from the second?

Solution # 1

It is easy to see that the two conducting planes can be replaced by three image charges as shown below



The force acting on the charge is

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0}\Big[\frac{1}{4a^2}\hat{e}_1 + \frac{1}{4a^2}\hat{e}_2 - \frac{1}{4c^2}(\hat{e}_1\cos\varphi + \hat{e}_2\sin\varphi)\Big]$$

where I've introduced notations $c = \sqrt{a^2+b^2}$ and $\varphi = \arctan\frac{y}{x}$. Let us bring the charge from infinity along the path shown above. The image charges will move with the original chagre according to symmetry so $\varphi = const$ along the path. The corresponding work is

$$W = \int_\infty^{a,b} \vec{F}\cdot\vec{dl} = -\frac{q^2}{4\pi\epsilon_0}\int_c^\infty dr\Big[\frac{1}{4r^2\cos\varphi} + \frac{1}{4r^2\sin\varphi} - \frac{1}{4r^2}\Big] = -\frac{q^2}{16\pi\epsilon_0}\Big[\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\Big]$$

Note that $W < 0$ as expected

Solution # 2

Use formula

$$W = \frac{1}{2}\sum_i q_i\phi(\vec{r}_i)$$

In our case, the induced charges do not contribute since the potential at the surface of the conductor is 0 so we get only the contribution from the point charge

$$W = \frac{1}{2}q\phi(\vec{r})$$

where $\phi(\vec{r})$ is the potential due to all other charges (= the potential due to image charges)

$$W = \frac{q}{2}\phi(\vec{r}) = \frac{q}{8\pi\epsilon_0}\Big[-\frac{q}{2a} - \frac{q}{2b} + \frac{q}{2c}\Big] = -\frac{q^2}{16\pi\epsilon_0}\Big[\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\Big]$$

Solution # 3

By symmetry, the work required to bring the charge to the conducting plane is equal to 1/4 of the work required to assemble all charges (real and image). The latter has the form

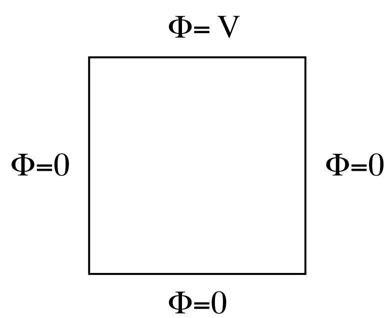
$$W_{\text{all charges}} = \frac{q^2}{4\pi\epsilon_0}\Big[-2\frac{q^2}{2a} - 2\frac{q^2}{2b} + 2\frac{q^2}{2c}\Big]$$

and therefore

$$W = \frac{1}{4}W_{\text{all charges}} = \frac{q^2}{16\pi\epsilon_0}\Big[-\frac{q^2}{a} - \frac{q^2}{b} + \frac{q^2}{c}\Big]$$

Problem 3 (5 points).

Find the solution of Laplace equation in a 2-dimensional square well of size a where three sides are kept at zero potential and the fourth side at constant potential V.



Solution

By analogy with the problem from Lecture 9

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sinh\alpha_n y \sin\alpha_n x, \qquad \alpha_n = \frac{\pi n}{a}$$

Boundary conditions at $x = 0, a$ and $y = 0$ are trivially satisfied. At $y = a$ we get

$$V = \sum_{n=1}^{\infty} A_n \sinh\alpha_n a \sin\alpha_n x$$

so

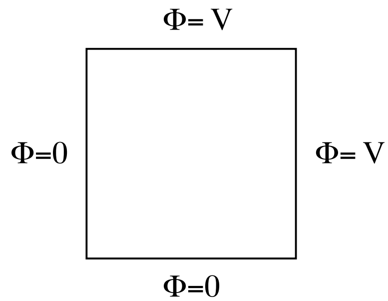
$$A_n = \frac{2V}{a}\sinh\pi n \int_0^a \sin\frac{\pi n x}{a} = \frac{2V}{a\sinh\pi n}\big[1 - (-1)^n\big]$$

and the solution takes the form

$$\Phi(x,y) = \frac{4V}{\pi}\sum_{n=\text{odd}} \frac{1}{n\sinh\pi n}\sin\frac{\pi n x}{a}\sinh\frac{\pi n y}{a}$$

Extra credit - 3 points.

Same for setup shown below



Solution

By superposition principle

$$\Phi(x,y) \;=\; \frac{4V}{\pi}\sum_{n=\text{odd}}\frac{1}{n\sinh\pi n}\sin\frac{\pi n x}{a}\sinh\frac{\pi n y}{a} \;+\; (x\leftrightarrow y)$$