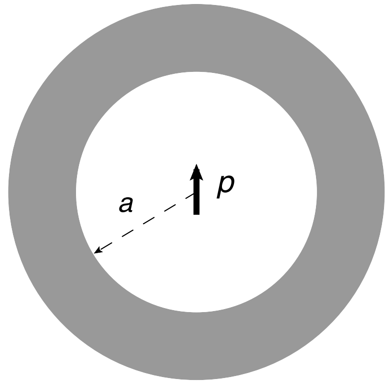
604 Midterm (16 points). Tue Nov 26, 10:55 a.m.-12:25 p.m.



Problem 1 (5 points).

Consider a spherical cavity of radius a cut out of a conducting metal. A pure dipole p is placed at the center of the cavity. Find the potential inside the cavity.

Solution

Assume $\vec{p} \parallel \hat{e}_3$. Due to azimuthal symmetry we can expand the potential in Legendre polynomials

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})$$

The potential is a sum of the potential of pure dipole

$$\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

and the potential of the induced charges on the conductor. At small r the potential due to the induced charges is $\sim \frac{Q_{\rm ind}}{4\pi\epsilon_0 a}$ $\sim$const so the singular behavior of the potential as $r \to 0$ is coincides with the dipole potential. The corresponding coefficients B_l are thus

$$B_1 = \frac{p}{4\pi\epsilon_0}, \quad B_{l\neq 1} = 0$$

and the expansion in Legendre polynomials takes the form

$$\Phi(r,\theta) = (A_1 r + \frac{p}{4\pi\epsilon_0 r^2})\cos\theta + \sum_{l\neq 1}^{\infty} A_l r^l P_l(\cos\theta)$$

From the boundary condition $\Phi(a,\theta) = 0$ and orthogonality of Legendre polynomials it is clear that $A_{l\neq 1} = 0$ and

$$A_1 a + \frac{p}{4\pi\epsilon_0 a^2} = 0 \qquad \Rightarrow A_1 = -\frac{p}{4\pi\epsilon_0 a^3}$$

The result for the potential is

$$\Phi(r,\theta) = \frac{p}{4\pi\epsilon_0}\Big(-\frac{r}{a^3} + \frac{1}{r^2}\Big)\cos\theta$$

Problem 2 (6 points).

Find the Dirichlet Green function of Laplace equation for the interior of infinite cylinder with radius a.

Solution

Up to Eqs. (3.37) and (3.38) from "Chapter 3" file everything is the same as for infinite space. The difference is in the boundary condition for $y_2(x')$. For infinite space, we had $y_2(x') \to 0$ as $x' \to \infty$ so the proper choice was $y_2(x') = K_m(x')$. Now, the boundary condition is $y_2(ka) = 0$ so we should take

$$y_2(x') \;=\; K_m(x') - \frac{K_m(ka)}{I_m(ka)} I_m(x')$$

The Wronskian $W(y_1(x'), y_2(x') = -\frac{1}{x'}$ is the same as for infinite space case since $W(I(x'), I(x') = 0$ so the Green function can be obtained from Eq. (3.7.45) by replacement of $K_m(ks)$ by

$$L_m(ks) \;=\; K(ks) - \frac{K_m(ka)}{I_m(ka)} I_m(ks)$$

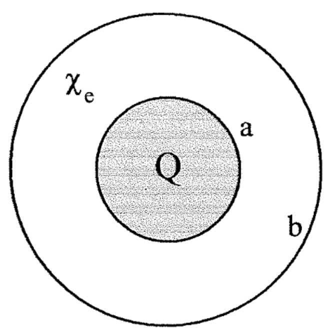
Finally, the Green function reads

$$G(\vec{r}, \vec{r'}) \;=\; \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\; e^{ik(z-z')} I_m(|k|s_<) L_m(|k|s_>)$$

A quick check at $a \to \infty$: we get Eq. (3.7.45) since the additional term in L vanishes due to $\frac{K_m(ka)}{I_m(ka)} \overset{a\to\infty}{\to} 0$.

Problem 1 (5 points).

A conducting sphere of radius a carries charge Q. It is surrounded by dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.



Solution

Due to spherical symmetry, the electric field and displacement are radial. Due to Gauss' law, the electric displacement at $r > a$ is

$$D(r) \;=\; \frac{Q}{4\pi r^2}$$

so the electric field is given by

$$E(r) \;=\; \frac{Q}{4\pi r^2 \epsilon}\theta(b-r)\theta(r-a) + \frac{Q}{4\pi r^2 \epsilon_0}\theta(r-b)$$

The energy of this setup is

$$W \;=\; \frac{1}{2}\int \vec{D}\cdot\vec{E} d^3x \;=\; 4\pi\frac{\epsilon}{2}\int_a^b dr\ r^2 E^2(r) + 4\pi\frac{\epsilon_0}{2}\int_b^\infty dr\ r^2 E^2(r) \;=\; \frac{Q}{8\pi\epsilon_0}\Big[\frac{1}{\chi_e}\big(\frac{1}{a}-\frac{1}{b}\big)+\frac{1}{b}\Big]$$