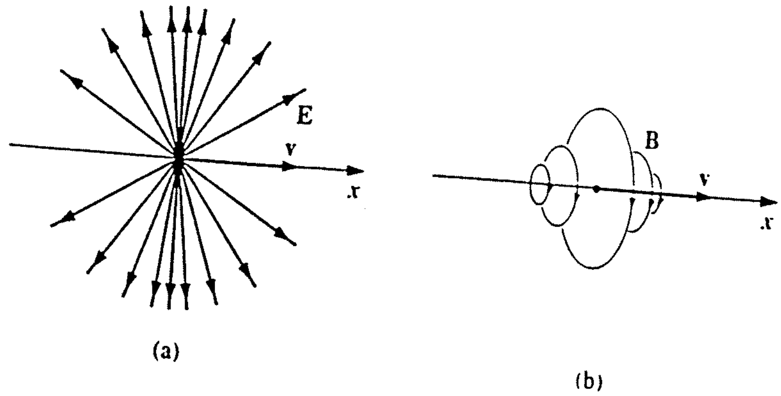
Conservation of momentum in electrodynamics

Electric and magnetic fields of a moving charge:



(a) (b)

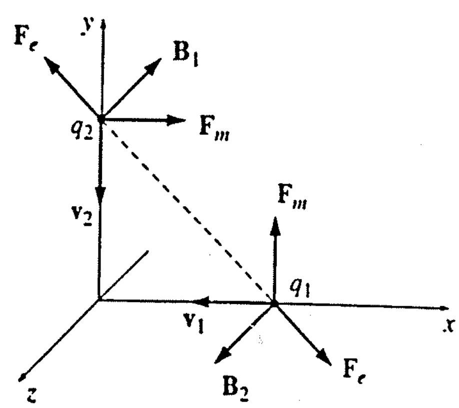
Test of Newton's Third Law:

$F_{electric}$ - OK

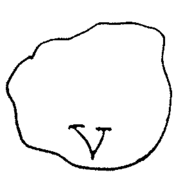
F_{mag} - not OK

Q: How?

A: Fields themselves carry momentum



Maxwell's stress tensor.



$\vec{F} = \int_V d^3x\, \rho(r)\,(\vec{E}(\vec{r}) + \vec{v}\times\vec{B}(r))$ ← Lorentz force acting on charges distributed over volume V

$\vec{f} = \rho\,(E + \vec{v}\times\vec{B})$ -

- force per unit volume

$\frac{d}{dt}\vec{P}_{mech} = \int_V d^3x\, \vec{f}(x)$ $\quad$ $\vec{P}_{mech}$ - total momentum of the particles in a volume V

A couple of mathematical tricks:

$$\vec{f} = \rho\vec{E} + \rho\vec{v}\times\vec{B} = \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{-\vec{\nabla}\times\vec{E}} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E}) \Rightarrow$$

$$\Rightarrow \vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\vec{E}^2) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\vec{E}\times(\vec{\nabla}\times\vec{E}) \Rightarrow \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

similarly, $\vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\vec{\nabla}(B^2) - (\vec{B}\cdot\vec{\nabla})\vec{B}$

$$\Rightarrow \vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{B}\cdot\vec{\nabla})\vec{B} + \underbrace{(\vec{\nabla}\cdot\vec{B})\vec{B}}_{0}\right] - \frac{1}{2}\vec{\nabla}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

we add this for symmetry

Maxwell strength tensor

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \qquad \overleftrightarrow{T} \equiv \{T_{ij}\}$$

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \mu_0\epsilon_0\frac{\partial\vec{S}}{\partial t}$$

$$f_j = \epsilon_0\,\partial_iE_i)E_j + (E_i\partial_i)E_j - \frac{\epsilon_0}{2}\partial_i(E^2) + \left(E\leftrightarrow B,\ \epsilon_0\leftrightarrow\frac{1}{\mu_0}\right) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})_j = \partial_iT_{ij} - \frac{1}{c^2}\frac{\partial S_j}{\partial t}$$

About tensors.

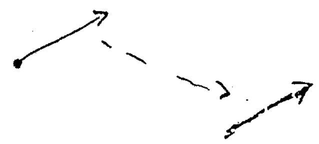
Let us at first recall the definition of a vector (for simplicity, in 2 dimensions)

vector ≡ "magnitude + direction"

Mathematically, vector is defined as a pair of numbers ("components" of the vector)

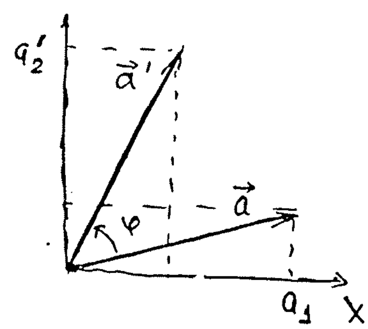
Not any pair of numbers constitutes a vector; this pair must behave in a specific way under rotations (and shifts)

How does a vector behave under translations and rotations
Under translations - trivial behavior



$a_1 \to a_1$
$a_2 \to a_2$

Under rotations - non-trivial transformation



(a_1', a_2') - components of the new vector $\vec{a}'$

$$a_1' = a_1 \cos\varphi - a_2 \sin\varphi$$
$$a_2' = a_1 \sin\varphi + a_2 \cos\varphi$$

(check: $a_1'^2 + a_2'^2 = a_1^2 + a_2^2 = \vec{a}^2$)

Mathematical notation

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \iff a_i' = \sum_{k=1}^{2} R_{ik} a_k$$

$R(\varphi)$ - matrix of the rotation on angle φ

$R_{11} = R_{22} = \cos\varphi$
$R_{21} = -R_{12} = \sin\varphi$

Definition: a vector is a pair of numbers (a_1, a_2) which behaves like $(a_1, a_2) \to (a_1', a_2')$ $\quad a_i' = R_{ik}(\varphi) a_k$ under the rotations.

Let us now take two vectors, $\vec{a}$ and $\vec{b}$, and consider the products of their components
$a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2$

These 4 numbers behave under rotations as follows:

$$a_1' b_1' = (a_1 \cos\varphi - a_2 \sin\varphi)(b_1 \cos\varphi - b_2 \sin\varphi)$$
$$a_1' b_2' = (a_1 \cos\varphi - a_2 \sin\varphi)(b_1 \sin\varphi + b_2 \cos\varphi)$$
$$a_2' b_1' = (a_1 \sin\varphi + a_2 \cos\varphi)(b_1 \cos\varphi - b_2 \sin\varphi)$$
$$a_2' b_2' = (a_1 \sin\varphi + a_2 \cos\varphi)(b_1 \sin\varphi + b_2 \cos\varphi)$$

Formally,
$$a_i' b_j' = \sum_{k,l} R_{ik} R_{jl} a_k b_l \quad (*)$$

Definition of a (2-dim) tensor:

4 numbers $t_{11}, t_{12}, t_{21}, t_{22}$ form a tensor if they behave as (*) under the rotations (and do not change under shifts)

$t'_{11}, t'_{12}, t'_{21}, t'_{22}$ - components of the rotated tensor

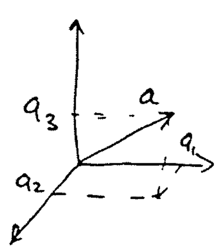
$$t'_{11} = t_{11}\cos^2\varphi - (t_{12}+t_{21})\sin\varphi\cos\varphi + t_{22}\sin^2\varphi$$
$$t'_{12} = t_{11}\cos\varphi\sin\varphi + t_{12}\cos^2\varphi - t_{21}\sin^2\varphi - t_{22}\cos\varphi\sin\varphi$$
$$t'_{21} = t_{11}\cos\varphi\sin\varphi - t_{12}\sin^2\varphi + t_{21}\cos^2\varphi - t_{22}\cos\varphi\sin\varphi$$
$$t'_{22} = t_{11}\sin^2\varphi + (t_{12}+t_{21})\sin\varphi\cos\varphi + t_{22}\cos^2\varphi$$

In matrix notations $t'_{ij} = \sum_{k,l} R_{ik} R_{jl} t_{jl}$

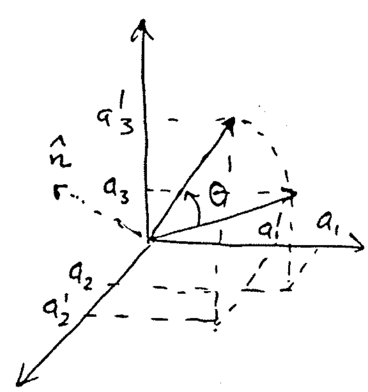
t_{12} is not necessarily equal to t_{21}; if $t_{12}=t_{21}$ → "symmetric tensor"

For 3-dimensional tensors - similar definition

3 d vector: a_1, a_2, a_3



Under rotations



$a'_i = \sum R_{ik} a_k$

matrix of 3d rotations

(can be parametrized by $\hat{n}, \theta$ or by 3 Euler angles)

⇒ 3d vector is a set of 3 numbers $\underbrace{}_{(a_1 a_2 a_3)}$ which transform as $a'_i = \sum_k R_{ik} a_k$ under the rotation

Similarly to the 2d case, a 3d tensor is defined as a set of 9 numbers $t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}$ which behave as products of components of two vectors under the rotations

$$t'_{ij} = \sum_{k,l=1}^{3} R_{ik} R_{jl} t_{kl}$$

Let us prove that δ_{ij} is a tensor (for simplicity, in the 2d case)

$$\delta'_{11} = \overset{=1}{\delta_{11}}\cos^2\varphi - (\underset{=\emptyset}{\delta_{12}}+\underset{=\emptyset}{\delta_{21}})\sin\varphi\cos\varphi + \overset{=1}{\delta_{22}}\sin^2\varphi = \cos^2\varphi+\sin^2\varphi = 1 = \delta_{11}$$

$$\delta'_{12} = \delta_{11}\cos\varphi\sin\varphi + \underset{=\emptyset}{\delta_{12}}\cos^2\varphi - \underset{=\emptyset}{\delta_{21}}\sin^2\varphi - \delta_{22}\cos\varphi\sin\varphi = \emptyset$$

$$\delta'_{21} = (\delta_{11}-\delta_{22})\sin\varphi\cos\varphi - {}_{12}\sin^2\varphi + {}_{21}\cos^2\varphi = \emptyset$$

$$\delta'_{22} = \delta_{11}\sin^2\varphi + (\delta_{12}+\delta_{21})\sin\varphi\cos\varphi + \delta_{22}\cos^2\varphi = 1 = \delta_{22}$$

$\Rightarrow$ δ_{ij} is a tensor. It can be proved for 3d case in a similar way

We will need two mathematical properties

1. A sum of two tensors is also a tensor $\left((A+B)_{ij} \overset{def}{\equiv} A_{ij}+B_{ij}\right)$

Proof: $A'_{ij} + B'_{ij} = \sum_{k,l} R_{ik}R_{jl}A + \sum_{k,l} R_{ik}R_{jl}B_{kl} = \sum_{k,l} R_{ik}R_{jl}(A_{kl}+B_{kl})$

$\Rightarrow (A+B)'_{ij} = \sum_{k,l} R_{ik}R_{jl}(A+B)_{kl}$

2. If a_i is a vector and t_{ij} is a tensor, the sums $\sum a_k t_{ki}$ form a vector (denoted by $(\vec a\cdot\overleftrightarrow{t})$)

Proof: $(\vec a\cdot\overleftrightarrow{t})'_i \equiv (\vec a'\cdot\overleftrightarrow{t}')_i = \sum_k a'_k t'_{ki} = \sum_k \sum_{l,m,n} R_{kl}a_l R_{km}R_{in}t_{mn}$

Property of the rotation matrix

$\sum_k R_{kl}R_{km} = \delta_{mn}$ (Proof: for any vector a $\vec a'^2 = a^2 \Rightarrow$
$\Rightarrow \sum_k a'_k a'_k = \sum_{k,l,m} \underbrace{R_{kl}R_{km}} a_l a_m = \sum_l a_l a_l = \sum_{l,m}\underbrace{\delta_{lm}} a_l a_m$)

$\Rightarrow (\vec a\cdot\overleftrightarrow{t})_i = \sum_{m,l,n}\delta_{ml}R_{in}a_l t_{mn} = \sum_{l,n} R_{in}a_l t_{ln} = \sum_n R_{in}(\sum a_l t_{ln}) = \sum_n R_{in}(\vec a\cdot\overleftrightarrow{t})_n$

$\Rightarrow$ $\vec a\cdot\overleftrightarrow{t}$ is a vector

Maxwell stress tensor

$$T_{ij} = \epsilon_0\left(\underset{\substack{\uparrow\\ \text{tensor}}}{E_iE_j} - \frac{1}{2}\underset{\substack{\uparrow\\ \text{tensor}}}{\delta_{ij}}E^2\right) + \frac{1}{\mu_0}\left(\underset{\substack{\uparrow\\ \text{tensor}}}{B_iB_j} - \frac{1}{2}\underset{\substack{\uparrow\\ \text{tensor}}}{\delta_{ij}}B^2\right)$$

$\Rightarrow$ T_{ij} is a tensor

$(\vec\nabla\cdot\overleftrightarrow{T})_i = \sum_k \partial_k T_{ki} = \sum_k \epsilon_0(\partial_k(E_kE_i) - \frac{1}{2}\partial_i(E^2)) + \frac{1}{\mu_0}(E\leftrightarrow B) = \sum_k \epsilon_0((\partial_k E_k)E_i +$
$+ E_k\partial_k E_i - \frac{1}{2}\partial_i E^2) + \frac{1}{\mu_0}(E\leftrightarrow B) \Rightarrow (\vec\nabla\cdot\overleftrightarrow{T})_i = \epsilon_0[(\vec\nabla\cdot\vec E)E_i + (\vec E\cdot\vec\nabla)E_i - \frac{1}{2}\partial_i E^2] +$
$\frac{1}{\mu_0}(E\leftrightarrow B) \Rightarrow \vec\nabla\cdot\overleftrightarrow{T} = \epsilon_0[(\vec\nabla\cdot\vec E)\vec E + (\vec E\cdot\vec\nabla)\vec E - \frac{1}{2}\vec\nabla E^2] + \frac{1}{\mu_0}[(\vec\nabla\cdot\vec B)\vec B + (\vec B\cdot\vec\nabla)\vec B - \frac{1}{2}\vec\nabla B^2]$

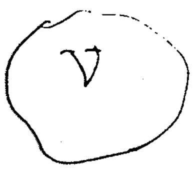
Maxwell stress tensor

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{\delta_{ij}}{2} B^2\right)$$

force (per unit volume) acting on a charged body

$$f = \vec{\nabla} \cdot \overleftrightarrow{T} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \qquad \left(\text{in components } f_i = \sum_j \partial_j T_{ji} - \epsilon_0 \mu_0 \frac{\partial S_i}{\partial t}\right)$$

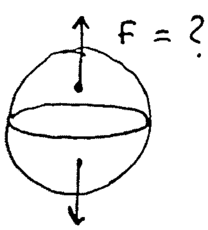
Total force on the charges in $\mathcal{V}$



$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \mu_0 \epsilon_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S}\, d^3x$$

↑ force per unit area (stress) acting on the surface

Example:



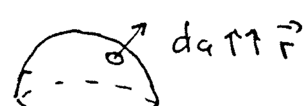
Uniformly charged sphere
The (electric) field is static ⇒

$$\Rightarrow \vec{F} = \int_{\text{Surface}} \overleftrightarrow{T} \cdot d\vec{a} = \int_{\text{"bowl"}} \overleftrightarrow{T} \cdot d\vec{a} + \int_{\text{"disk"}} \overleftrightarrow{T} \cdot d\vec{a}$$

1. $\int \overleftrightarrow{T} \cdot d\vec{a}$ From symmetry it is clear that $\vec{F} \uparrow\uparrow e_3 \Rightarrow$ it is enough to calculate $\int (\overleftrightarrow{T} \cdot d\vec{a})_3$

$$F_3 = \int_{\text{bowl}} \sum_{j=1}^{3} \epsilon_0 \left(E_3 E_j - \frac{\delta_{3j}}{2} E^2\right) da_j \qquad d\vec{a} = \hat{r}\, da \Rightarrow da_j = \hat{r}_j\, da$$

$$\vec{E} = \frac{Q}{4\pi R^2 \epsilon_0} \hat{r} \Rightarrow E_j = \frac{Q}{4\pi R^2 \epsilon_0} r_j$$



$$\Rightarrow \sum_j \left(E_3 E_j - \frac{\delta_{3j}}{2} E^2\right) da_j = \left(\frac{Q}{4\pi R^2 \epsilon_0}\right)^2 \sum_j \underbrace{\left(\hat{r}_3 \hat{r}_j - \frac{\delta_{3j}}{2}\right) \hat{r}_j}_{\hat{r}_3 - \frac{\hat{r}_3}{2} = \frac{\hat{r}_3}{2}} da =$$

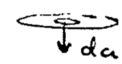
$$F_3 = \frac{Q^2}{32\pi^2 R^4 \epsilon_0^2} \int_{\text{bowl}} da\, \hat{r}_3 =$$

$$= \frac{Q^2}{32\pi^2 R^4 \epsilon_0^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, R^2 \sin\theta \cos\theta = \frac{Q^2}{16\pi R^2 \epsilon_0} \underbrace{\int_0^{\pi/2} d\theta \sin\theta \cos\theta}_{1/2} = \frac{Q^2}{32\pi R^2 \epsilon_0}$$

2. $\oint \overleftrightarrow{T} \cdot d\vec{a}$ Again,

$$F_3 = \int_{\text{disc}} \sum_j \epsilon_0 \left(E_3 E_j - \frac{\delta_{3j}}{2} E^2\right) da_j$$

For the disc $d\vec{a} = -\hat{e}_3\, da$



the field inside the uniformly charged sphere is

$$\vec{E} = \frac{Q r}{4\pi\epsilon_0 R^3}\hat{r}$$

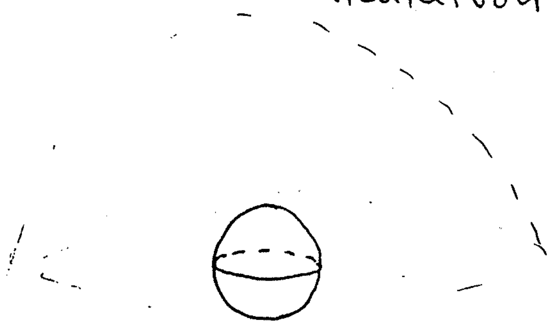
$$\Rightarrow F_3 = -\int \sum_j \epsilon_0 \left(\hat{r}_3 \hat{r}_3 - \frac{\delta_{33}}{2}\right)\left(\frac{Qr}{4\pi R^3 \epsilon_0}\right)^2 da =$$

Inside the disc $\vec{r} \perp \hat{e}_3 \Rightarrow r_3 = \emptyset$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi R^3 \epsilon_0}\right)^2 \int r^2 da = \frac{Q^2\, 2\pi}{32\pi^2 R^6 \epsilon_0}\int_0^R r^3 dr =$$

$$= \frac{Q^2}{64\pi\epsilon_0 R^2} \quad \Rightarrow \vec{F}_{\text{total}} = \vec{F}_{\text{bowl}} + \vec{F}_{\text{disc}} = \frac{3Q^4}{64\pi\epsilon_0 R^2}\hat{e}_3$$

Alternative calculation



surface = xy plane + ∞ hemisphere

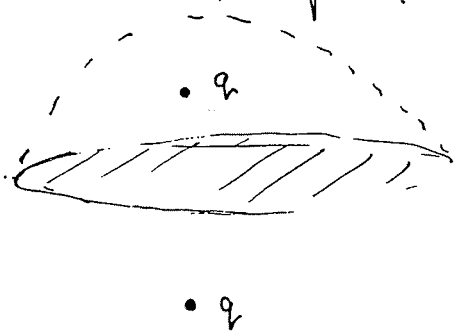
$$\int_{\text{surface}} \overleftrightarrow{T} \cdot d\vec{a} = \int_{\text{xy plane}} \overleftrightarrow{T} \cdot d\vec{a} + \cancel{\int_{\infty\,\text{hemisphere}} \overleftrightarrow{T} \cdot d\vec{a}}$$

$$\Rightarrow F_3 = -\int_{\substack{\text{all}\\ \text{plane}}} \sum_j \epsilon_0 \left(\hat{r}_3 \hat{r}_3 - \frac{\delta_{33}}{2}\right) E^2(r)\, da \Rightarrow$$

$$\Rightarrow F_3 = \frac{1}{2}\int_{\text{all plane}} \epsilon_0 E^2(r)\, da = \frac{\cancel{2}\pi}{\cancel{2}}\int_0^R r\,dr\, \epsilon_0 \left(\frac{Qr}{4\pi\epsilon_0 R^3}\right)^2 + \frac{\cancel{2}\pi}{\cancel{2}}\int_R^\infty r\,dr\, \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r}\right)^2$$

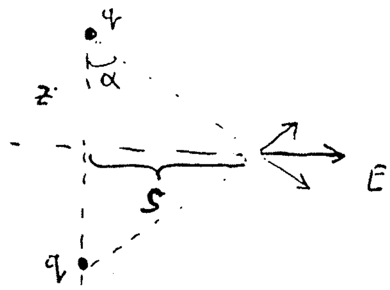
$$= \frac{Q^2}{16\pi\epsilon_0}\left(\int_0^R \frac{1}{R^6} r^3 dr + \int_R^\infty \frac{dr}{r^3}\right) = \frac{Q^2}{16\pi\epsilon_0}\left(\frac{1}{4R^2} + \frac{1}{2R^2}\right) = \frac{3Q^2}{64\pi\epsilon_0 R^2}$$

Another example: Coulomb law



$$F_3 = \int_{\text{xy plane}} \overleftrightarrow{T} \cdot d\vec{a} =$$

$$= -\int_{\text{plane}} da \left(E_3 E_3 - \frac{1}{2}E^2\right)\epsilon_0$$



$$\vec{E} = 2\,\frac{q}{(z^2+r^2)}\,\frac{\sin\alpha}{4\pi\epsilon_0}\,\hat{s} \Rightarrow E_3 = \emptyset$$

$$E^2 = \frac{q^2}{4\pi^2\epsilon_0^2}\,\frac{z^2}{(z^2+s^2)^3}$$

$$\Rightarrow F_3 = \frac{\epsilon_0}{2}\int_0^\infty s\,ds\int_0^{2\pi} d\varphi\,\frac{q^2}{4\pi^2\epsilon_0^2}\,\frac{z^2}{(z^2+s^2)^3} = \frac{z^2q^2}{4\pi\epsilon_0}\int_0^\infty ds\,\frac{s}{(z^2+s^2)^3} = \frac{z^2q^2}{4\pi\epsilon_0}\,\frac{1}{4z^4}$$

$$\Rightarrow F_3 = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{2z}\right)^2 \equiv \text{Coulomb law.}$$

Conservation of momentum

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} \Rightarrow \frac{d\vec{p}_{mech}}{dt} = -\mu_0\epsilon_0\frac{d}{dt}\int_V \vec{S}\,d^3x + \oint_S \overleftrightarrow{T}\cdot d\vec{a}$$

$\vec{P}_{em} = \mu_0\epsilon_0\int_V \vec{S}\,d^3x$ → momentum stored in the electromagnetic fields themselves

$$\Rightarrow \frac{d}{dt}(\vec{P}_{mech}+\vec{P}_{em}) = \oint_S \overleftrightarrow{T}\cdot da \Rightarrow \frac{d}{dt}(\vec{P}_{mech}+\vec{P}_{em})_{\text{all space}} = \emptyset$$

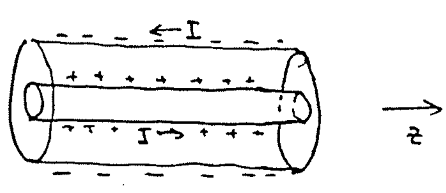
Conservation of momentum in a differential form

$$\vec{P}_{mech} = \int_V d^3x\,\vec{\wp}_{mech} \qquad \vec{P}_{em} = \int_V d^3x\,\vec{\wp}_{em} \qquad \vec{\wp}_{em} = \mu_0\epsilon_0\vec{S}$$

$$\Rightarrow \frac{d}{dt}\int_V d^3x\,(\vec{\wp}_{mech}+\vec{\wp}_{em}) = \oint_S \overleftrightarrow{T}\cdot d\vec{a} = \int \vec{\nabla}\cdot\overleftrightarrow{T}\,d^3x \Rightarrow \frac{d}{dt}(\wp_{mech}+\wp_{em}) = \vec{\nabla}\cdot\overleftrightarrow{T}$$

$\overleftrightarrow{T}$ is the momentum flux density

Example



$E = B = \emptyset$ outside the cable

$E = \frac{\lambda}{2\pi\epsilon_0 s}\,\hat{s} \qquad B = \frac{\mu_0 I}{2\pi s}\,\hat{\varphi}$ inside

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{e}_3$$

$$\Rightarrow \vec{P}_{e.m.} = \mu_0\epsilon_0\int\vec{S}\,d^3x = \frac{\mu_0\lambda I\hat{e}_3}{4\pi^2}\int_0^\ell dz\int_a^b\frac{s\,ds}{s^2}\,2\pi = \frac{\mu_0\lambda I\ell}{2\pi}\,\ln\frac{b}{a}\,\hat{e}_3$$

This momentum is compensated by the "hidden" mechanical momentum (a relativistic effect)

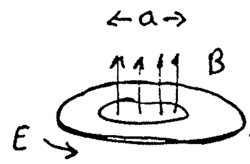
Angular momentum of electromagnetic fields

$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ – energy (per unit volume)

$\vec{\mathcal{P}}_{em} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 \vec{E} \times \vec{B}$ – momentum

$\Rightarrow \quad \ell_{em} = \vec{r} \times \vec{\mathcal{P}} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$ – angular momentum of electromagnetic fields

Example

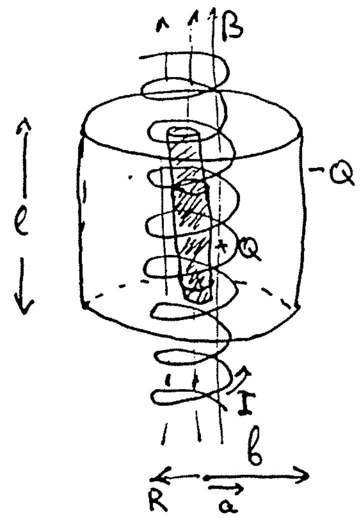


← charged loop

$dN = d\ell\, \vec{R} \times \vec{F} = R \lambda E d\ell$, $\quad R\vec{E}dQ$

B is switched off ⇒ electric field ⇒ torque $\quad N = -B\lambda a^2 \frac{dB}{dt} \Rightarrow$

⇒ total angular momentum is $\int N dt = \lambda \pi a^2 b B_0$



$$E = \frac{Q/\ell}{2\pi\epsilon_0 s}\hat{s} \qquad a<s<b$$
$$E = \phi \qquad \text{otherwise}$$
} before switching off the current

$$B = \mu_0 n I \hat{e}_3\, \Theta(R-s)$$

$\Theta(R-s) = \phi$ outside the solenoid

$$\Rightarrow \vec{\mathcal{P}}_{em} = \epsilon_0 \vec{E} \times \vec{B} = -\frac{\mu_0 n I Q}{2\pi \ell s}\hat{\varphi}$$

$\hat{e}_3 \times \hat{\varphi} = -\hat{s}$
$\hat{s} \times \hat{\varphi} = \hat{e}_3$

$$\Rightarrow \vec{\ell}_{em} = \vec{r} \times \vec{\mathcal{P}}_{em} = (\vec{s} + z\hat{e}_3) \times \left(-\frac{\mu_0 n I Q}{2\pi\ell s}\right)\hat{\varphi}$$

$$= -\frac{\mu_0 n I Q}{2\pi\ell}\hat{e}_3 + \underbrace{z\frac{\mu_0 n I}{2\pi\ell s}\hat{r}}_{\text{will average to } \phi \text{ due to cylindrical symmetry}} = -\frac{\mu_0 n I Q}{2\pi\ell}\hat{e}_3$$

$$\Rightarrow \vec{L}_{em} = \vec{\ell}_{em} \cdot \overbrace{\text{volume}}^{\pi(R^2-a^2)\ell} =$$
$$= -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\ell$$

When the current is turned off

$$\vec{E} = \begin{cases} -\frac{\mu_0 n}{2}\frac{dI}{dt}\frac{R^2}{s}\hat{\varphi} & s>R \\ -\frac{\mu_0 n}{2}\frac{dI}{dt} s\hat{\varphi} & s<R \end{cases}$$

$\oint E\, d\ell = E\, 2\pi s = -\pi R^2 \mu_0 n \frac{\partial I}{\partial t}$

⇒ the torque on the outer cylider is

$$\vec{N}_b = \hat{r} \times (-Q\vec{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\vec{e}_3$$

$\Rightarrow$ The outer cylinder picks up the angular momentum

$$\vec{L}_b = \int N_b\, dt = \frac{1}{2}\mu_0 n Q R^2 \hat{e}_3 \int_0^\infty \frac{dI}{dt}\, dt = -\frac{\mu_0 n}{2} I Q R^2 \hat{e}_3$$

Similarly, the torque on the inner cylinder is

$$\vec{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\, \hat{e}_3$$

$\Rightarrow$ $L_a = \frac{1}{2}\mu_0 n I Q a^2 \hat{e}_3$

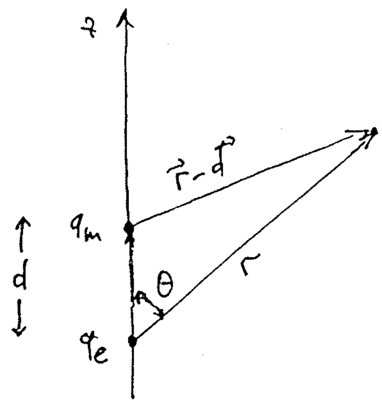
Check: $L_b + L_a = -\frac{1}{2}\mu_0 n I Q^2 (R^2 - a^2)\, \ell$

Thomson's dipole

$$\underset{\substack{\text{(electric}\\ \text{charge)}}}{\overset{\bullet}{q_e}} \qquad \overset{\bullet}{q_m} \leftarrow \begin{array}{l}\text{magnetic}\\ \text{monopole}\end{array}$$

Let us calculate the angular momentum stored in the electromagnetic field of this "dipole"

$$l = \epsilon_0 \, \vec r \times (\vec E(r) \times \vec B(\vec r))$$



$$l(\vec r) = \epsilon_0 \vec r \times \left(\frac{q_e}{4\pi\epsilon_0} \frac{\vec r}{r^3} \times \frac{q_m \mu_0}{4\pi} \frac{\vec r - \vec d}{|\vec r - \vec d|^3} \right)$$

$$\left(\text{NB}: \quad \widehat{a-b} \neq \hat a - \hat b \quad \Rightarrow \quad \frac{\vec r - \vec r'}{|r-r'|^3} \neq \frac{\hat r - \hat r'}{|r-r'|^2} \right)$$

$$l(\vec r) = \frac{q_e q_m}{16\pi^2} \mu_0 \frac{-\vec r \times (\vec r \times \vec d)}{r^3 (r^2 + d^2 - 2dr\cos\theta)^{3/2}}$$

$$\vec r \times (\vec r \times \vec d) = \vec r (\vec d \cdot \vec r) - \vec d r^2 = -d r^2 (1-\cos\theta) \hat e_3$$

$$\Rightarrow \vec L = \int d^3x \, l(\vec r) = \int_0^\infty r^2 dr \int_0^\pi d\theta \sin\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{q_e q_m}{16\pi^2} \mu_0 \frac{d(1-\cos\theta)\hat e_3}{r(d^2+r^2-2dr\cos\theta)^{3/2}}$$

$$= \frac{q_e q_m}{8\pi} \mu_0 d \int_0^\infty dr \int_0^2 dt \frac{t r \hat e_3}{((d-r)^2 + 2drt)^{3/2}} = \qquad t \equiv 1-\cos\theta$$

$$= \frac{q_e q_m}{8\pi} \mu_0 \underbrace{\int_0^\infty d\lambda \int_0^2 dt \frac{t\lambda \hat e_3}{((\lambda-1)^2 + 2\lambda t)^{3/2}}}_{=2} = \frac{\mu_0}{4\pi} q_e q_m \hat e_3$$

Calculation: $\displaystyle\int_0^\infty d\lambda \int_0^2 dt \frac{\lambda t}{((\lambda-1)^2+2\lambda t)^{3/2}} = \int_0^\infty d\lambda \int_0^2 dt \frac{\lambda}{|\lambda-1|^3} \frac{t}{(1+\frac{2\lambda}{(\lambda-1)^2} t)^{3/2}} =$

$$\boxed{\frac{2\lambda}{(\lambda-1)^2} t \equiv y} = \int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \int_0^{2\frac{2\lambda}{(\lambda-1)^2}} \frac{y}{(1+y)^{3/2}} dy = \int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \int_1^{\frac{(\lambda+1)^2}{(\lambda-1)^2}} dy \frac{y-1}{y^{3/2}} =$$

$$\int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \cdot 2\left(\frac{\lambda+1}{|\lambda-1|} - 1 - 1 + \frac{|\lambda-1|}{\lambda+1} \right) = \int_0^\infty d\lambda \frac{1}{2\lambda} \left(\lambda + 1 - 2|\lambda-1| + \frac{(\lambda-1)^2}{\lambda+1} \right) = \int_0^\infty \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - |\lambda-1| \right) =$$

$$\int_0^1 \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - 1 + \lambda \right) + \int_1^\infty \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - \lambda + 1 \right) = \int_0^1 d\lambda \frac{2\lambda}{\lambda+1} + \int_1^\infty \frac{d\lambda}{\lambda} \frac{2}{\lambda+1} =$$

$$= 2\int_0^1 d\lambda \frac{\lambda}{\lambda+1} + 2\int_0^1 d\lambda' \frac{1}{1+\lambda'} = 2 \Big)$$

$\Rightarrow L = \frac{\mu_0}{4\pi} q_e q_m$ $\leftarrow$ does not depend on separation

Quant. mech: $L = n\frac{\hbar}{2}$ $\quad n = 0, 1, 2 \ldots$

$\Rightarrow \frac{\mu_0}{4\pi} q_e q_m = n\frac{\hbar}{2}$ if the magnetic monopole would exist somewhere