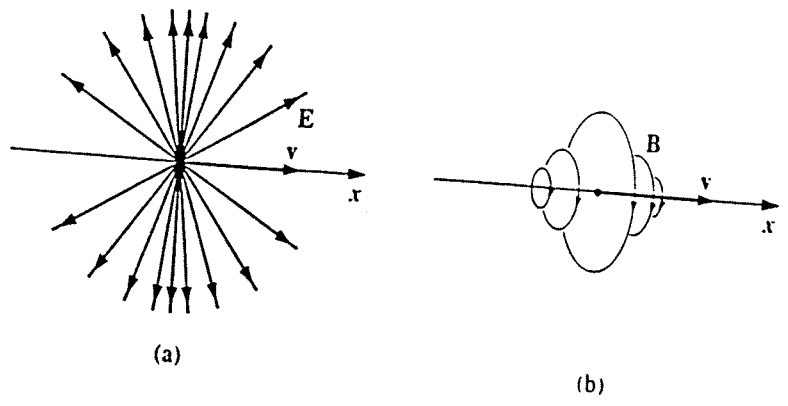


Conservation of momentum in electrodynamics

Electric and magnetic fields of a moving charge:

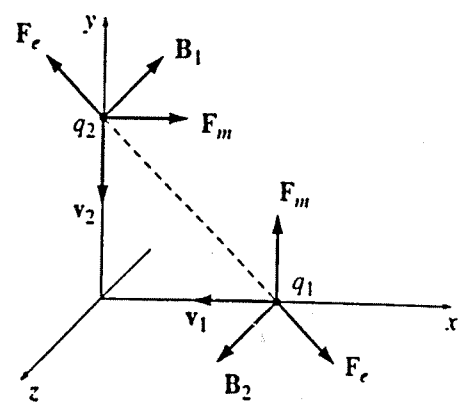


Test of Newton's Third Law:

Felectric - OK
 Fmag - not OK

Q: How?

A: Fields themselves carry momentum



Maxwell's stress tensor.



$$\vec{F} = \int_V d^3x \rho(r) (\vec{E}(r) + \vec{v} \times \vec{B}(r))$$

← Lorentz force acting on charges distributed over volume V

$$\vec{f} = \rho (\vec{E} + \vec{v} \times \vec{B}) -$$

- force per unit volume

$$\frac{d}{dt} \vec{P}_{mech} = \int_V d^3x \vec{f}(x)$$

\vec{P}_{mech} - total momentum of the particles in a volume V

A couple of mathematical tricks:

$$\vec{f} = \rho \vec{E} + \rho \vec{\nabla} \times \vec{B} = \rho \vec{E} + \vec{J} \times \vec{B} = \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

$$\frac{\partial \vec{f}}{\partial t} \times \vec{B} = \frac{d}{dt} (\vec{E} \times \vec{B}) - \vec{E} \times \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\vec{\nabla} \times \vec{E}) \Rightarrow$$

$$\Rightarrow \vec{f} = \epsilon_0 [(\vec{\nabla} \cdot \vec{E}) \vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E})] - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

$$\vec{\nabla} (\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A} \Rightarrow$$

$$\Rightarrow \vec{\nabla} (E^2) = 2(\vec{E} \cdot \vec{\nabla}) \vec{E} + 2\vec{E} \times (\vec{\nabla} \times \vec{E}) \Rightarrow \vec{E} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{2} \vec{\nabla} (E^2) - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

similarly, $\vec{B} \times (\vec{\nabla} \times \vec{B}) = \frac{1}{2} \vec{\nabla} (B^2) - (\vec{B} \cdot \vec{\nabla}) \vec{B}$

$$\Rightarrow \vec{f} = \epsilon_0 [(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E}] + \frac{1}{\mu_0} [(\vec{B} \cdot \vec{\nabla}) \vec{B} + (\vec{\nabla} \cdot \vec{B}) \vec{B}] - \frac{1}{2} \vec{\nabla} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

we add this for symmetry

Maxwell's strength tensor

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2) \quad \vec{T} \equiv \{T_{ij}\}$$

$$\vec{f} = \vec{\nabla} \cdot \vec{T} - \mu_0 \epsilon_0 \frac{\partial \vec{S}}{\partial t}$$

$$f_j = \epsilon_0 (\partial_i E_i) E_j + (E_i \partial_i) E_j - \frac{\epsilon_0}{2} \partial_i (E^2) + (E \leftrightarrow B, \epsilon_0 \leftrightarrow \frac{1}{\mu_0}) - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})_j = \partial_i T_{ij} - \frac{1}{c^2} \frac{\partial S_j}{\partial t}$$

About tensors.

Let us at first recall the definition of a vector (for simplicity, in 2 dimensions)

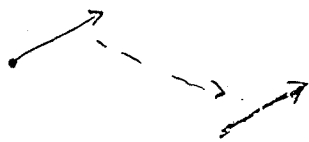


vector \equiv "magnitude + direction"

Mathematically, vector is defined as a pair of numbers ("components" of the vector)

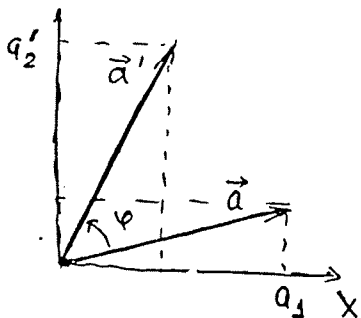
Not any pair of numbers constitutes a vector; this pair must behave in a specific way under rotations (and shifts)

How does a vector behave under translations and rotations
 Under translations - trivial behavior



$$\begin{aligned} a_1 &\rightarrow a_1 \\ a_2 &\rightarrow a_2 \end{aligned}$$

Under rotations - non-trivial transformation



(a'_1, a'_2) - components of the new vector \vec{a}'

$$\begin{aligned} a'_1 &= a_1 \cos \varphi - a_2 \sin \varphi \\ a'_2 &= a_1 \sin \varphi + a_2 \cos \varphi \end{aligned}$$

(check: $a_1'^2 + a_2'^2 = a_1^2 + a_2^2 = \vec{a}^2$)

Mathematical notation

$$\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Leftrightarrow a'_i = \sum_{k=1}^2 R_{ik} a_k$$

$R(\varphi)$ - matrix of the rotation on angle φ

$$\begin{aligned} R_{11} &= R_{22} = \cos \varphi \\ R_{21} &= -R_{12} = \sin \varphi \end{aligned}$$

Definition: a vector is a pair of numbers (a_1, a_2) which behaves like $(a_1, a_2) \rightarrow (a'_1, a'_2)$ under the rotations. $a'_i = R_{ik}(\varphi) a_k$

Let us now take two vectors, \vec{a} and \vec{b} , and consider the products of their components $a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2$

These 4 numbers behave under rotations as follows:

$$\begin{aligned} a'_1 b'_1 &= (a_1 \cos \varphi - a_2 \sin \varphi)(b_1 \cos \varphi - b_2 \sin \varphi) \\ a'_1 b'_2 &= (a_1 \cos \varphi - a_2 \sin \varphi)(b_1 \sin \varphi + b_2 \cos \varphi) \\ a'_2 b'_1 &= (a_1 \sin \varphi + a_2 \cos \varphi)(b_1 \cos \varphi - b_2 \sin \varphi) \\ a'_2 b'_2 &= (a_1 \sin \varphi + a_2 \cos \varphi)(b_1 \sin \varphi + b_2 \cos \varphi) \end{aligned}$$

Formally,

$$a'_i b'_j = \sum_{k,l} R_{ik} R_{jl} a_k b_l \quad (*)$$

Definition of a (2-dim) tensor: -4-

4 numbers $t_{11}, t_{12}, t_{21}, t_{22}$ form a tensor if they behave as (*) under the rotations (and do not change under shifts)

$t'_{11}, t'_{12}, t'_{21}, t'_{22}$ - components of the rotated tensor

$$t'_{11} = t_{11} \cos^2 \varphi - (t_{12} + t_{21}) \sin \varphi \cos \varphi + t_{22} \sin^2 \varphi$$

$$t'_{12} = t_{11} \cos \varphi \sin \varphi + t_{12} \cos^2 \varphi - t_{21} \sin^2 \varphi - t_{22} \cos \varphi \sin \varphi$$

$$t'_{21} = t_{11} \cos \varphi \sin \varphi - t_{12} \sin^2 \varphi + t_{21} \cos^2 \varphi - t_{22} \cos \varphi \sin \varphi$$

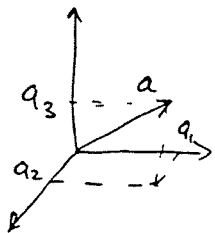
$$t'_{22} = t_{11} \sin^2 \varphi + (t_{12} + t_{21}) \sin \varphi \cos \varphi + t_{22} \cos^2 \varphi$$

In matrix notations $t'_{ij} = \sum_{k,l=1}^2 R_{ik} R_{jl} t_{kl}$

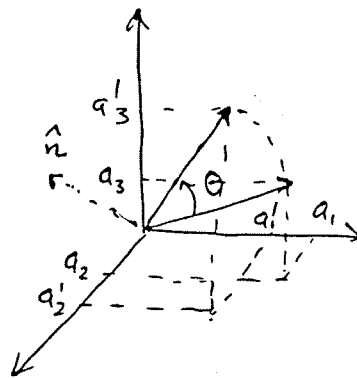
t_{12} is not necessarily equal to t_{21} ; if $t_{12} = t_{21} \rightarrow$ "symmetric tensor"

For 3-dimensional tensors - similar definition

3 d vector: a_1, a_2, a_3



Under rotations



$$a'_i = \sum_k R_{ik} a_k$$

matrix of 3d rotations

(can be parametrized by \hat{n}, θ or by 3 Euler angles)

\Rightarrow 3d vector is a set of 3 numbers (a_1, a_2, a_3) which transform as $a'_i = \sum_k R_{ik} a_k$ under the rotation

Similarly to the 2d case, a 3d tensor is defined as a set of 9 numbers

$$t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}$$

which behave as products of components of two vectors under the rotations

$$t'_{ij} = \sum_{k,l=1}^3 R_{ik} R_{jl} t_{kl}$$

Let us prove that δ_{ij} is a tensor (for simplicity, in the 2d case)

$$\delta'_{11} = \delta_{11} \cos^2 \varphi - (\delta_{12} + \delta_{21}) \sin \varphi \cos \varphi + \delta_{22} \sin^2 \varphi = \cos^2 \varphi + \sin^2 \varphi = 1 = \delta_{11}$$

$$\delta'_{12} = \delta_{11} \cos \varphi \sin \varphi + \delta_{12} \cos^2 \varphi - \delta_{21} \sin^2 \varphi - \delta_{22} \cos \varphi \sin \varphi = 0$$

$$\delta'_{21} = (\delta_{11} - \delta_{22}) \sin \varphi \cos \varphi - \delta_{12} \sin^2 \varphi + \delta_{21} \cos^2 \varphi = 0$$

$$\delta'_{22} = \delta_{11} \sin^2 \varphi + (\delta_{12} + \delta_{21}) \sin \varphi \cos \varphi + \delta_{22} \cos^2 \varphi = 1 = \delta_{22}$$

$\Rightarrow \delta_{ij}$ is a tensor. It can be proved for 3d case in a similar way

We will need two mathematical properties

1. A sum of two tensors is also a tensor ($(A+B)_{ij} \stackrel{\text{def}}{=} A_{ij} + B_{ij}$)

Proof: $A'_{ij} + B'_{ij} = \sum_{k,l} R_{ik} R_{jl} A_{kl} + \sum_{k,l} R_{ik} R_{jl} B_{kl} = \sum_{k,l} R_{ik} R_{jl} (A_{kl} + B_{kl})$
 $\Rightarrow (A+B)'_{ij} = \sum_{k,l} R_{ik} R_{jl} (A+B)_{kl}$

2. If a_i is a vector and t_{ij} is a tensor, the sums $\sum a_k t_{ki}$ form a vector (denoted by $(\vec{a} \cdot \vec{T})$)

Proof: $(\vec{a} \cdot \vec{T})'_i = (\vec{a}' \cdot \vec{T}')_i = \sum_k a'_k t'_{ki} = \sum_k \sum_{l,m,n} R_{kl} a_l R_{km} R_{in} t_{mn}$

Property of the rotation matrix

$\sum_k R_{kl} R_{km} = \delta_{lm}$ (Proof: for any vector a $a'^2 = a^2 \Rightarrow \sum_k a'_k a'_k = \sum_{k,l,m} R_{kl} R_{km} a_l a_m = \sum_l a_l a_l = \sum_{l,m} \delta_{lm} a_l a_m$)

$\Rightarrow (\vec{a} \cdot \vec{T})'_i = \sum_{m,l,n} \delta_{ml} R_{in} a_l t_{mn} = \sum_{l,n} R_{in} a_l t_{ln} = \sum_n R_{in} (\sum_l a_l t_{ln}) = \sum_n R_{in} (\vec{a} \cdot \vec{T})_n$

$\Rightarrow \vec{a} \cdot \vec{T}$ is a vector

Maxwell stress tensor

$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$
 (Note: $E_i E_j$ and $B_i B_j$ are tensors, δ_{ij} is a tensor) $\Rightarrow T_{ij}$ is a tensor

$(\vec{\nabla} \cdot \vec{T})_i = \sum_k \partial_k T_{ki} = \sum_k \epsilon_0 (\partial_k (E_k E_i) - \frac{1}{2} \partial_i (E^2)) + \frac{1}{\mu_0} (E \leftrightarrow B) = \sum_k \epsilon_0 (\partial_k E_k) E_i + E_k \partial_k E_i - \frac{1}{2} \partial_i E^2 + \frac{1}{\mu_0} (E \leftrightarrow B) \Rightarrow (\vec{\nabla} \cdot \vec{T})_i = \epsilon_0 [(\vec{\nabla} \cdot \vec{E}) E_i + (\vec{E} \cdot \vec{\nabla}) E_i - \frac{1}{2} \partial_i E^2] + \frac{1}{\mu_0} (E \leftrightarrow B) \Rightarrow \vec{\nabla} \cdot \vec{T} = \epsilon_0 [(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} - \frac{1}{2} \vec{\nabla} E^2] + \frac{1}{\mu_0} [(\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} - \frac{1}{2} \vec{\nabla} B^2]$

Maxwell stress tensor

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{\delta_{ij}}{2} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{\delta_{ij}}{2} B^2)$$

force (per unit volume) acting on a charged body

$$\vec{f} = \vec{\nabla} \cdot \vec{T} - \epsilon_0 \mu_0 \frac{\partial \vec{s}}{\partial t} \quad (\text{in components } f_i = \sum_j \partial_j T_{ji} - \epsilon_0 \mu_0 \frac{\partial s_i}{\partial t})$$

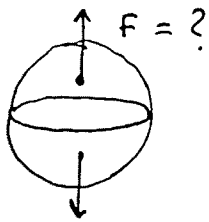
Total force on the charges in V



$$\vec{F} = \oint_S \vec{T} \cdot d\vec{a} - \mu_0 \epsilon_0 \frac{d}{dt} \int_V \vec{s} d^3x$$

force per unit area (stress) acting on the surface

Example:



Uniformly charged sphere

The (electric) field is

static \Rightarrow

$$\Rightarrow \vec{F} = \int_{\text{Surface}} \vec{T} \cdot d\vec{a} = \int_{\text{"bowl"}} \vec{T} \cdot d\vec{a} + \int_{\text{"disk"}} \vec{T} \cdot d\vec{a}$$

1. $\int_{\text{bowl}} \vec{T} \cdot d\vec{a}$ From symmetry it is clear that $\vec{F} \uparrow \uparrow \hat{e}_3 \Rightarrow$

it is enough to calculate $\int (\vec{T} \cdot d\vec{a})_3$

$$F_3 = \int_{\text{bowl}} \sum_{j=1}^3 \epsilon_0 (E_3 E_j - \frac{\delta_{3j}}{2} E^2) da_j \quad d\vec{a} = \hat{r} da \Rightarrow da_j = \hat{r}_j da$$

$$\vec{E} = \frac{Q}{4\pi R^2 \epsilon_0} \hat{r} \Rightarrow E_j = \frac{Q}{4\pi R^2 \epsilon_0} \hat{r}_j$$



$$\Rightarrow \sum_j (E_3 E_j - \frac{\delta_{3j}}{2} E^2) da_j = \left(\frac{Q}{4\pi R^2 \epsilon_0} \right)^2 \sum_j \underbrace{(\hat{r}_3 \hat{r}_j - \frac{\delta_{3j}}{2})}_{\hat{r}_3 - \frac{\hat{r}_3}{2} = \frac{\hat{r}_3}{2}} \hat{r}_j da =$$

$$F_3 = \frac{Q^2}{32\pi^2 R^4 \epsilon_0^2} \int_{\text{bowl}} da \hat{r}_3 =$$

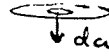
$$= \frac{Q^2}{32\pi^2 R^4 \epsilon_0^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi R^2 \sin\theta \cos\theta = \frac{Q^2}{16\pi R^2 \epsilon_0} \int_0^{\pi/2} \underbrace{d\theta \sin\theta \cos\theta}_{1/2} = \frac{Q^2}{32\pi R^2 \epsilon_0}$$

2. $\int \vec{T} \cdot d\vec{a}$

Again,

$$F_3 = \int_{\text{disc}} \sum_j \epsilon_0 (E_3 E_j - \frac{\delta_{3j}}{2} E^2) da_j$$

For the disc $d\vec{a} = -\hat{e}_3 da$



The field inside the uniformly charged sphere is

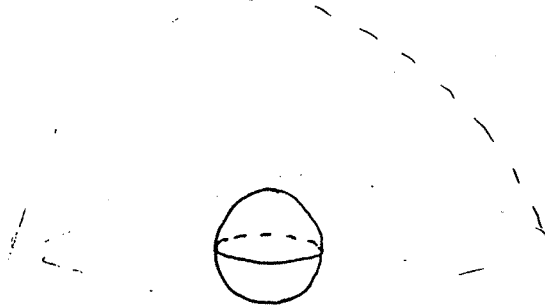
$$\vec{E} = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{r}$$

$$\begin{aligned} \Rightarrow F_3 &= - \int \sum_j \epsilon_0 (\hat{r}_3 \hat{r}_j - \frac{\delta_{3j}}{2}) \left(\frac{Qr}{4\pi R^3 \epsilon_0}\right)^2 da = \\ &= \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi R^3 \epsilon_0}\right)^2 \int r^2 da = \frac{Q^2}{32\pi^2 R^6 \epsilon_0} \int_0^R r^3 dr = \\ &= \frac{Q^2}{64\pi\epsilon_0 R^2} \end{aligned}$$

Inside the disc
 $\vec{r} \perp \hat{e}_3 \Rightarrow r_3 = 0$

$$\Rightarrow \vec{F}_{\text{total}} = \vec{F}_{\text{bowl}} + \vec{F}_{\text{disc}} = \frac{3Q^2}{64\pi\epsilon_0 R^2} \hat{e}_3$$

Alternative calculation



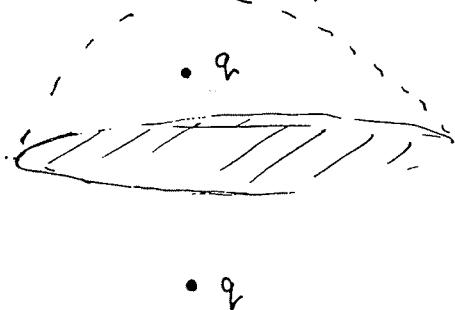
surface = xy plane + ∞ hemisphere

$$\int_{\text{surface}} \vec{T} \cdot d\vec{a} = \int_{\text{xy plane}} \vec{T} \cdot d\vec{a} + \cancel{\int_{\infty \text{ hemisphere}} \vec{T} \cdot d\vec{a}}$$

$$\Rightarrow F_3 = - \int \sum_j \epsilon_0 (\hat{r}_3 \hat{r}_j - \frac{\delta_{3j}}{2}) E^2(r) da \Rightarrow$$

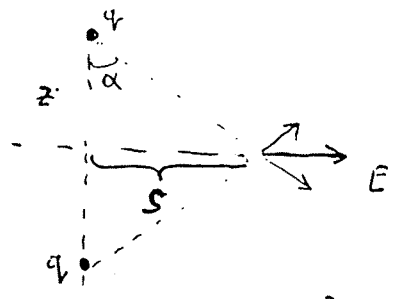
$$\begin{aligned} \Rightarrow F_3 &= \frac{1}{2} \int_{\text{all plane}} \epsilon_0 E^2(r) da = \frac{2\pi}{2} \int_0^R r dr \epsilon_0 \left(\frac{Qr}{4\pi\epsilon_0 R^3}\right)^2 + \frac{2\pi}{2} \int_R^\infty r dr \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r}\right)^2 \\ &= \frac{Q^2}{16\pi\epsilon_0} \left(\int_0^R \frac{1}{R^6} r^3 dr + \int_R^\infty \frac{dr}{r^3} \right) = \frac{Q^2}{16\pi\epsilon_0} \left(\frac{1}{4R^2} + \frac{1}{2R^2} \right) = \frac{3Q^2}{64\pi\epsilon_0 R^2} \end{aligned}$$

Another example: Coulomb law



$$F_3 = \int_{\text{xy plane}} \vec{T} \cdot d\vec{a} =$$

$$= - \int_{\text{plane}} da (E_3 E_3 - \frac{1}{2} E^2) \epsilon_0$$



$$\vec{E} = 2 \frac{q}{(z^2 + s^2)} \frac{d\Omega}{4\pi\epsilon_0} \hat{s} \Rightarrow E_z = 0$$

$$E^2 = \frac{q^2}{4\pi^2\epsilon_0^2} \frac{z^2}{(z^2 + s^2)^3}$$

$$\Rightarrow F_z = \frac{\epsilon_0}{2} \int_0^\infty s ds \int_0^{2\pi} d\varphi \frac{q^2}{4\pi^2\epsilon_0^2} \frac{z^2}{(z^2 + s^2)^3} = \frac{z^2 q^2}{4\pi\epsilon_0} \int_0^\infty ds \frac{s}{(z^2 + s^2)^3} = \frac{z^2 q^2}{4\pi\epsilon_0} \frac{1}{4z^4}$$

$$\Rightarrow F_z = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{2z}\right)^2 \equiv \text{Coulomb law.}$$

Conservation of momentum

$$\vec{F} = \frac{d\vec{p}_{\text{mech}}}{dt} \Rightarrow \frac{d\vec{p}_{\text{mech}}}{dt} = -\mu_0\epsilon_0 \frac{d}{dt} \int_V \vec{S} d^3x + \oint_S \vec{T} \cdot d\vec{a}$$

$\vec{p}_{\text{em}} = \mu_0\epsilon_0 \int_V \vec{S} d^3x$ → momentum stored in the electromagnetic fields themselves

$$\Rightarrow \frac{d}{dt} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_S \vec{T} \cdot d\vec{a} \Rightarrow \frac{d}{dt} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}})_{\text{all space}} = 0$$

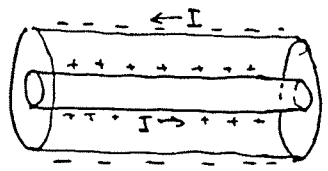
Conservation of momentum in a differential form

$$\vec{p}_{\text{mech}} = \int_V d^3x \vec{p}_{\text{mech}} \quad \vec{p}_{\text{em}} = \int_V d^3x \vec{p}_{\text{em}} \quad \vec{p}_{\text{em}} = \mu_0\epsilon_0 \vec{S}$$

$$\Rightarrow \frac{d}{dt} \int_V d^3x (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_S \vec{T} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{T} d^3x \Rightarrow \frac{d}{dt} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \vec{\nabla} \cdot \vec{T}$$

\vec{T} is the momentum flux density

Example



$E = B = 0$ outside the cable

$$E = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \quad B = \frac{\mu_0 I}{2\pi s} \hat{\varphi} \quad \text{inside}$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2} \hat{e}_z$$

$$\Rightarrow \vec{p}_{\text{e.m.}} = \mu_0\epsilon_0 \int \vec{S} d^3x = \frac{\mu_0 \lambda I \hat{e}_z}{4\pi^2} \int_0^l dz \int_a^b \frac{s ds}{s^2} 2\pi = \frac{\mu_0 \lambda I l}{2\pi} \ln \frac{b}{a} \hat{e}_z$$

This momentum is compensated by the "hidden" mechanical momentum (a relativistic effect)

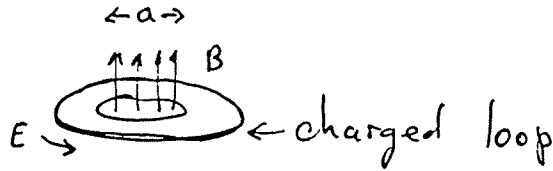
Angular momentum of electromagnetic fields

$u_{em} = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$ - energy (per unit volume)

$\vec{D}_{em} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 \vec{E} \times \vec{B}$ - momentum

$\Rightarrow \vec{L}_{em} = \vec{r} \times \vec{D} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$ - angular momentum of electromagnetic fields

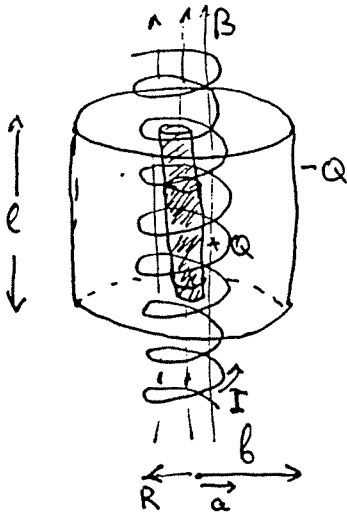
Example



$dN = dl \vec{r} \times \vec{F} = R \int E d$
 $R \int E dQ$

B is switched off \Rightarrow electric field \Rightarrow torque
 \Rightarrow total angular momentum is $S N dt = \lambda \pi a^2 b B_0$

$N = -B \lambda a^2 \frac{dB}{dt} \Rightarrow$



$E = \frac{Q/l}{2\pi\epsilon_0 s} \hat{s} \quad a < s < b$
 $E = \phi$ otherwise } before switching off the current

$B = \mu_0 n I \hat{e}_3 \Theta(R-s)$

$\hat{e}_3 \times \hat{\phi} = -\hat{s}$
 $\hat{s} \times \hat{\phi} = \hat{e}_3$
 $\hat{\phi} = \phi$ outside the solenoid

$\Rightarrow \vec{D}_{em} = \epsilon_0 \vec{E} \times \vec{B} = - \frac{\mu_0 n I Q}{2\pi l s} \hat{\phi}$

$\Rightarrow \vec{L}_{em} = \vec{r} \times \vec{D}_{em} = (\vec{s} + z \hat{e}_3) \times (- \frac{\mu_0 n I Q}{2\pi l s}) \hat{\phi}$

$= - \frac{\mu_0 n I Q}{2\pi l} \hat{e}_3 + z \frac{\mu_0 n I}{2\pi l s} \hat{r} = - \frac{\mu_0 n I Q}{2\pi l} \hat{e}_3$

will average to ϕ due to cylindrical symmetry

$\Rightarrow \vec{L}_{em} = \vec{L}_{em} \cdot \text{volume} = \underbrace{\pi(R^2 - a^2)}_{\text{volume}} \cdot (- \frac{1}{2} \mu_0 n I Q (R^2 - a^2) \hat{e}_3)$

When the current is turned off

$\vec{E} = \begin{cases} - \frac{\mu_0 n}{2} \frac{dI}{dt} \frac{R^2}{s} \hat{\phi} & s > R \\ - \frac{\mu_0 n}{2} \frac{dI}{dt} s \hat{\phi} & s < R \end{cases}$

$\int E dl = E 2\pi s = - \pi R^2 \mu_0 n \frac{\partial I}{\partial t}$

\Rightarrow the torque on the outer cylinder is

$\vec{N}_b = \hat{r} \times (-Q \vec{E}) = \frac{1}{2} \mu_0 n Q R^2 \frac{dI}{dt} \hat{e}_3$

⇒ The outer cylinder picks up the angular momentum

$$\vec{L}_b = \int N_b dt = \frac{1}{2} \mu_0 n Q R^2 \hat{e}_3 \int \frac{dI}{dt} dt = -\frac{\mu_0 n}{2} I Q R^2 \hat{e}_3$$

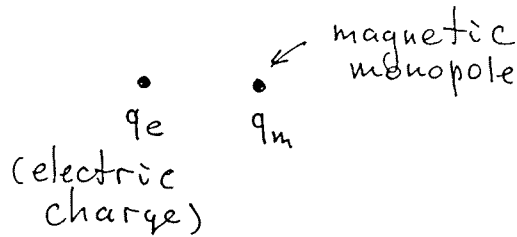
Similarly, the torque on the inner cylinder is

$$\vec{N}_a = -\frac{1}{2} \mu_0 n Q a^2 \frac{dI}{dt} \hat{e}_3$$

$$\Rightarrow L_a = \frac{1}{2} \mu_0 n I Q a^2 \hat{e}_3$$

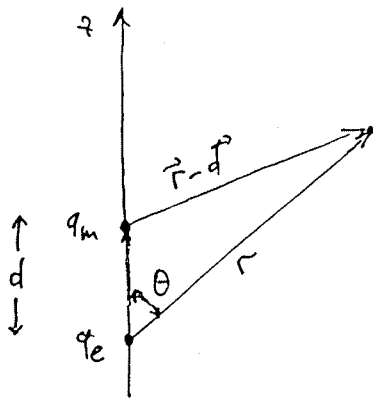
$$\text{Check: } L_b + L_a = -\frac{1}{2} \mu_0 n I Q^2 (R^2 - a^2) \hat{e}_3$$

Thomson's dipole



Let us calculate the angular momentum stored in the electromagnetic field of this "dipole"

$$l = \epsilon_0 \vec{r} \times (\vec{E}(\vec{r}) \times \vec{B}(\vec{r}))$$



$$l(\vec{r}) = \epsilon_0 \vec{r} \times \left(\frac{q_e}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \times \frac{q_m \mu_0}{4\pi} \frac{\vec{r}-\vec{d}}{|\vec{r}-\vec{d}|^3} \right)$$

(NB: $\hat{a}-\hat{b} \neq \hat{a}-\hat{b} \Rightarrow \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \neq \frac{\hat{r}-\hat{r}'}{|\vec{r}-\vec{r}'|^2}$)

$$l(\vec{r}) = \frac{q_e q_m}{8\pi} \mu_0 \frac{-\vec{r} \times (\vec{r} \times \vec{d})}{r^3 (r^2 + d^2 - 2dr \cos\theta)^{3/2}}$$

$$\vec{r} \times (\vec{r} \times \vec{d}) = \vec{r}(\vec{d} \cdot \vec{r}) - \vec{d}r^2 = -dr^2(1-\cos\theta) \hat{e}_3$$

$$\begin{aligned} \Rightarrow \vec{L} &= \int d^3x l(\vec{r}) = \int_0^\infty r^2 dr \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \frac{q_e q_m \mu_0}{16\pi^2} \frac{d(1-\cos\theta) \hat{e}_3}{r(d^2+r^2-2dr\cos\theta)^{3/2}} \\ &= \frac{q_e q_m \mu_0 d}{8\pi} \int_0^\infty dr \int_0^\pi d\theta \frac{t r \hat{e}_3}{((d-r)^2+2dr t)^{3/2}} \quad t \equiv 1-\cos\theta \\ &= \frac{q_e q_m \mu_0}{8\pi} \int_0^\infty d\lambda \int_0^\pi d\theta \frac{t \lambda \hat{e}_3}{(\lambda^2-1+2\lambda t)^{3/2}} = \frac{\mu_0}{4\pi} q_e q_m \hat{e}_3 \end{aligned}$$

Calculation: $\int_0^\infty d\lambda \int_0^\pi d\theta \frac{\lambda t}{(\lambda^2-1+2\lambda t)^{3/2}} = \int_0^\infty d\lambda \int_0^\pi d\theta \frac{\lambda}{(\lambda^2-1)^{3/2}} \frac{t}{(1+\frac{2\lambda}{\lambda^2-1}t)^{3/2}} =$

$$\frac{2\lambda}{\lambda^2-1} t \equiv y \quad \int_0^\infty d\lambda \frac{\lambda-1}{4\lambda} \int_0^{2\frac{\lambda-1}{\lambda+1}} \frac{y}{(1+y)^{3/2}} dy = \int_0^\infty d\lambda \frac{\lambda-1}{4\lambda} \int_{\frac{\lambda-1}{\lambda+1}}^1 dy \frac{y-1}{y^{3/2}} =$$

$$\int_0^\infty d\lambda \frac{\lambda-1}{4\lambda} \cdot 2 \left(\frac{\lambda+1}{\lambda-1} - 1 - 1 + \frac{\lambda-1}{\lambda+1} \right) = \int_0^\infty d\lambda \frac{1}{2\lambda} \left(\lambda+1 - 2|\lambda-1| + \frac{\lambda-1}{\lambda+1} \right) = \int_0^\infty \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - |\lambda-1| \right) =$$

$$\int_0^1 \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - 1 + \lambda \right) + \int_1^\infty \frac{d\lambda}{\lambda} \left(\frac{\lambda^2+1}{\lambda+1} - \lambda + 1 \right) = \int_0^1 \frac{d\lambda}{\lambda} \frac{2\lambda}{\lambda+1} + \int_1^\infty \frac{d\lambda}{\lambda} \frac{2}{\lambda+1} =$$

$$= 2 \int_0^1 \frac{d\lambda}{\lambda} \frac{\lambda}{\lambda+1} + 2 \int_0^1 \frac{d\lambda'}{1+\lambda'} = 2$$

$\Rightarrow L = \frac{\mu_0}{4\pi} q_e q_m$ ← does not depend on separation

Quant. mech: $L = n \frac{\hbar}{2} \quad n = 0, 1, 2, \dots$

$\Rightarrow \frac{\mu_0}{4\pi} q_e q_m = n \frac{\hbar}{2}$ if the magnetic monopole would exist somewhere