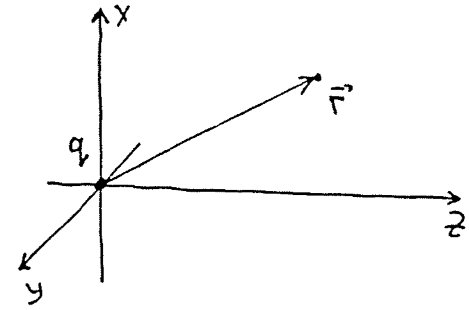
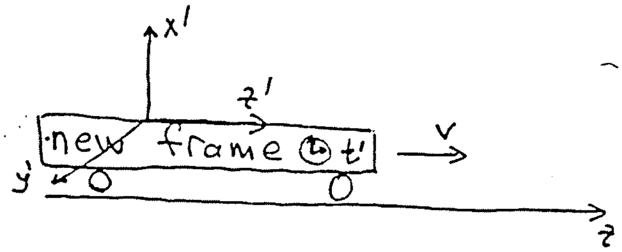
Electric field of a moving charge = Lorentz contraction of the Coulomb field



$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \quad \text{- Coulomb field}$$

Let us get into a new frame moving with speed v to the right ($\vec{v} = v e_3$)



In the new frame $\quad \left(\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}\right)$

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}z\right) \end{aligned}\right\} \quad \begin{aligned} z &= \gamma(z' + vt') \\ t &= \gamma\left(t' + \frac{v}{c^2}z'\right) \end{aligned} \qquad (*) \quad \text{Lorentz transformations}$$

In addition (to be demonstrated later)

$E'_{\parallel} = E_{\parallel}$ $\qquad \vec{E}' \equiv$ electric field in the new frame

$E'_{\perp} = \gamma E_{\perp}$ $\qquad$ (component of the electric field orthogonal to $\vec{v}$ is enhanced by $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$)

In our case

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3}{r^3} = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2}{r^3}}_{E_\perp} + \underbrace{\frac{q}{4\pi\epsilon_0}\frac{z\hat{e}_3}{r^3}}_{E_\parallel} \Rightarrow$$

$$\Rightarrow \vec{E}' = \gamma\frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2}{r^3} + \frac{q}{4\pi\epsilon_0}\frac{z\hat{e}_3}{r^3}$$

This is $\vec{E}'(\vec{r}, t)$ and we need $\vec{E}'(\vec{r}', t')$ $\Rightarrow$ use $(*)$ to express r in terms of $\vec{r}'$

$$\vec{E}' = \frac{q}{4\pi\epsilon_0}\gamma\frac{x'\hat{e}_1 + y'\hat{e}_2}{r^3} + \frac{q}{4\pi\epsilon_0}\frac{\gamma(z'+vt')\hat{e}_3}{r^3} = \frac{q\gamma}{4\pi\epsilon_0 r^3}(\underbrace{x'\hat{e}_1+y'\hat{e}_2+z'\hat{e}_3}_{\vec{r}'} + vt'\hat{e}_3)$$

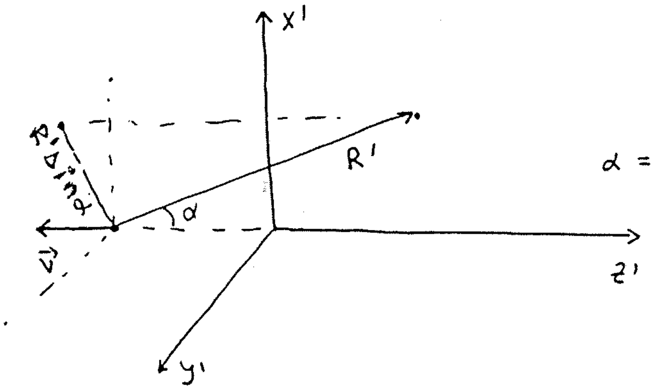
$$= \frac{q\gamma}{4\pi\epsilon_0}\frac{\vec{r}' + vt'\hat{e}_3}{r^3}$$

In the new frame, the charge q is moving with speed v to the left $\Rightarrow$ $\vec{R}' = \vec{r}' + vt'\hat{e}_3 = \vec{r}' - (-v\hat{e}_3)t'$ is the distance from the observation point to the position of the particle

$$\vec{E}(\vec{r}', t) = \frac{q\gamma}{4\pi\epsilon_0}\cdot\frac{\vec{R}'}{r^3}$$

Now, we should express $r^2 = x^2 + y^2 + z^2$ in terms of x', y', z', t'

$$r^2 = x'^2 + y'^2 + \gamma^2(z'+vt')^2 = \gamma^2\left((x'^2+y'^2)(1-\tfrac{v^2}{c^2}) + (z'+vt')^2\right) = \gamma^2(x'^2+y'^2+(z'+vt')^2 - \tfrac{v^2}{c^2}(x'^2+y'^2)) = \gamma^2(R'^2 - \tfrac{v^2}{c^2}R'^2\sin^2\theta)$$

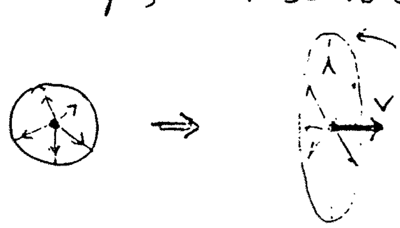


$\alpha = \pi - \theta \Rightarrow x'^2 + y'^2 = R'^2\sin^2\alpha = R'^2\sin^2\theta$

$$\Rightarrow \vec{E}'(\vec{r}', t') = \frac{q\gamma}{4\pi\epsilon_0}\frac{\vec{R}'}{\gamma^3 R'^3}\frac{1}{(\sqrt{1-v^2/c^2\sin^2\theta})^3} = \frac{q}{4\pi\epsilon_0}\frac{\vec{R}'}{R'^3}\frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}}$$

$\vec{E}(\vec{r})$ was $\uparrow\uparrow \vec{r}$, $\vec{E}'(\vec{R}', t')$ is also $\uparrow\uparrow R'$: $E_\parallel$ (E_z) gets an extra factor γ from the transformation of coordinates whereas $E_\perp$ (E_x and E_y) pick up their factors γ from the transformation of the field.

The electric (and magnetic) field of a rapidly moving charges resembles a pancake



$\theta = \frac{\pi}{2}$ $E' = \frac{E}{\sqrt{1-v^2/c^2}}$ - enhanced

$\theta = 0$ or π $E' = E(1-\frac{v^2}{c^2})$ - reduced