Contents

Chapter 7

Plane Electromagnetic Waves and Wave Propagation

7.1 Preliminaries

We begin by considering the propagation of waves in a non-conducting medium. Thus $\mathbf{J} \equiv 0$, we assume $\rho \equiv 0$ and Maxwell's equations reduce to

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= 0 \,, \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \,, \\
\nabla \cdot \mathbf{D} &= 0 \,, \\
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0 \,.
\end{aligned}$$

In the case of plane waves, it is sufficient to consider those propagating with a definite frequency ω, and hence time dependence $\exp\{-i\omega t\}$; essentially this is equivalent to taking the Fourier Transform. We have a set of linear, homogeneous equations and hence *all* fields have the same harmonic behaviour. Thus we may write Maxwell's equations as

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= 0 \,, \\
\nabla \cdot \mathbf{D} &= 0 \,, \\
\nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0 \,, \\
\nabla \times \mathbf{H} + i\omega \mathbf{D} &= 0 \,.
\end{aligned} \tag{7.1.1}$$

We will now specialize to the case of a linear constitutive relation between the fields: $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. We will also assume ϵ, μ are **real**. Note that later we will consider the complex case; taking them to be real corresponds to there being no energy losses. Then the last two equations of Eq. (7.1.1) become

$$\begin{aligned} \nabla \times \mathbf{E} - i\omega\mathbf{B} &= 0 \, , \\ \nabla \times \mathbf{B} + i\omega\epsilon\mu\mathbf{E} &= 0 \, . \end{aligned}$$

The two coupled first-order differential equations for **E** and **B** can be converted into two separate second-order differential equations for **E** and **B**. Indeed, using the first equation above to write $\mathbf{B} = \nabla \times \mathbf{E}/(i\omega)$ and substituting this result into the second equation, we get

$$\nabla \times (\nabla \times \mathbf{E})/(i\omega) + i\omega\epsilon\mu\mathbf{E} = 0 \, ,$$

or

$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2\epsilon\mu\mathbf{E} = 0 \, .$$

In the same way, substituting $\mathbf{E} = -\nabla \times \mathbf{B}/(i\omega\epsilon\mu)$ into $\nabla \times \mathbf{E} - i\omega\mathbf{B} = 0$ gives

$$\nabla \times (\nabla \times \mathbf{B}) - \omega^2\epsilon\mu\mathbf{B} = 0 \, .$$

Using that $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$, and similarly for **B**, and also incorporating $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, we get

$$\boxed{\begin{aligned} \nabla^2\mathbf{E} + \omega^2\epsilon\mu\mathbf{E} = 0 \, , \\ \nabla^2\mathbf{B} + \omega^2\epsilon\mu\mathbf{B} = 0 \, . \qquad (7.1.2) \end{aligned}}$$

These are known as the **Helmholtz wave equations**. As is well known, they support the plane-wave solutions

$$\boxed{\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{E_0} \\ \mathbf{B_0} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \quad (7.1.3)}$$

where $k = \omega\sqrt{\mu\epsilon}$. The ratio

$$v = \omega/k = 1/\sqrt{\mu\epsilon} \tag{7.1.4}$$

is the **phase velocity**.

We now recall that the velocity of light in the vacuum is given by

$$c = 1/\sqrt{\mu_0 \epsilon_0}. \tag{7.1.5}$$

Thus we can write

$$v = c/n \tag{7.1.6}$$

where

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0 \epsilon_0}} \tag{7.1.7}$$

is the **index of refraction**. It is usually a function of the frequency (recall color separation by a prism), and therefore the *phase velocity* is likewise *frequency dependent* - hence the name.

7.2 Propagation of Monochromatic Plane Wave

We will now consider in greater detail a monochromatic plane wave of frequency ω, propagating in the direction $\mathbf{n}$ with wave number k. Note that complex $\mathbf{n}$ corresponds to dissipation. We have seen that the solution of the Helmholtz equations are

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0 e^{ik\mathbf{n}\cdot\mathbf{x} - i\omega t} \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0 e^{ik\mathbf{n}\cdot\mathbf{x} - i\omega t} \end{aligned} \tag{7.2.1}$$

with

$$k^2 = \mu\epsilon\omega^2. \tag{7.2.2}$$

This is actually shorthand for

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}\left\{\mathbf{E_0} e^{ik\mathbf{n}\cdot\mathbf{x} - i\omega t}\right\}. \tag{7.2.3}$$

The imaginary part contains no physical information. It is important to remember this when considering quantities that are quadratic or higher in the fields, such as the energy density.

7.2.1 Energy Density for Monochromatic Plane Wave

Recall the expression for the energy density

$$u = \frac{1}{2}\left[\epsilon \mathbf{E}^2 + \frac{1}{\mu}\mathbf{B}^2\right]. \tag{7.2.4}$$

The *real* parts of the fields **B** and **E** must be taken **before** evaluating the quadratic terms. In the case of the *time-averaged* energy density, we have the particularly simple result

$$\langle u\rangle = \frac{1}{4}\left[\epsilon\mathbf{E}\cdot\mathbf{E}^* + \frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}^*\right] \tag{7.2.5}$$

where we use $\langle\ldots\rangle$ to denote that the time average has been taken, and the additional factor of one half arises from the observation

$$\langle\cos^2\omega t\rangle = 1/2. \tag{7.2.6}$$

Likewise, the time-averaged Poynting vector is

$$\langle\mathbf{S}\rangle = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* \tag{7.2.7}$$

This quantity is called the intensity of the wave.

7.3 Polarization of a Monochromatic Plane Wave

Applying $\nabla\cdot\mathbf{B} = 0$ and $\nabla\cdot\mathbf{E} = 0$ to the solutions

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \mathbf{E}_0 e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\\ \mathbf{B}(\mathbf{x},t) &= \mathbf{B}_0 e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\end{aligned} \tag{7.3.1}$$

of the Helmholtz equations, we find

$$\begin{aligned}\mathbf{n}\cdot\mathbf{B}_0 &= 0\ ,\\ \mathbf{n}\cdot\mathbf{E}_0 &= 0\ .\end{aligned} \tag{7.3.2}$$

Thus both **E** and **B** are perpendicular to the direction of propagation. We say that we have a **transverse wave**.

We now apply the remaining Maxwell equations

$$\begin{aligned}\nabla\times\mathbf{E} - i\omega\mathbf{B} &= 0\\ \nabla\times\mathbf{B} + i\omega\mu\epsilon\mathbf{E} &= 0,\end{aligned}$$

to yield

$$\mathbf{B}_0 = \frac{1}{i\omega}ik\,\mathbf{n}\times\mathbf{E}_0 = \sqrt{\mu\epsilon}\,\mathbf{n}\times\mathbf{E}_0 = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0. \tag{7.3.3}$$

Setting $c = 1/\sqrt{\mu\epsilon}$ to be the velocity of light *in the medium*, we see that both $c\mathbf{B}$ and **E** have the same magnitude, $\mathbf{B}_0 = \mathbf{n}\times\mathbf{E}_0/c$. We also have

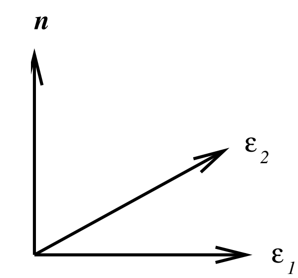
$$\mathbf{E}_0 = -\frac{1}{i\omega\mu\epsilon}ik\,\mathbf{n}\times\mathbf{B}_0 = -\frac{1}{\sqrt{\mu\epsilon}}\,\mathbf{n}\times\mathbf{B}_0 = -c\,\mathbf{n}\times\mathbf{B}_0. \tag{7.3.4}$$

$\mathcal{NB}$: Had we chosen to work with $\mathbf{H} = \mathbf{B}/\mu$, rather than $\mathbf{B}$, then we would have

$$\mathbf{H}_0 = \sqrt{\frac{\epsilon}{\mu}}\,\mathbf{n} \times \mathbf{E}_0 \equiv \frac{\mathbf{n} \times \mathbf{E}_0}{Z} \tag{7.3.5}$$

where $Z = \sqrt{\mu/\epsilon}$ is the **impedance**

We will now specialize to the case where $\mathbf{n}$ is indeed **real**. Then $\mathbf{B}_0$ is perpendicular to $\mathbf{E}_0$, and has the same phase.



The vectors $\mathbf{E}, \mathbf{B}$ and $\mathbf{n}$ form an orthogonal triad, and it is usual to introduce three mutually-orthogonal basis vectors $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ and $\mathbf{n}$ and to write the electromagnetic field as

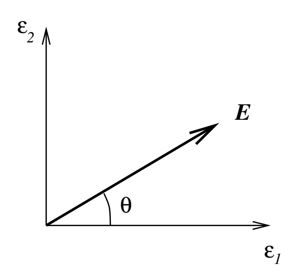
$$\begin{aligned} \mathbf{E_1}(\mathbf{x}, t) &= \boldsymbol{\epsilon}_1 E_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad ; \quad c\mathbf{B_1} = \boldsymbol{\epsilon}_2 E_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \mathbf{E_2}(\mathbf{x}, t) &= \boldsymbol{\epsilon}_2 E_2 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad ; \quad c\mathbf{B_2} = -\boldsymbol{\epsilon}_1 E_2 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \end{aligned} \tag{7.3.6}$$

Note that E_1 and E_2 can be complex to take into account a phase shift between the two plane waves.

The general solution for the wave equation is

$$\mathbf{E}(\mathbf{x}, t) = (\boldsymbol{\epsilon}_1 E_1 + \boldsymbol{\epsilon}_2 E_2) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{7.3.7}$$

Linear Polarization



If E_1 and E_2 have the **same phase** we talk about a **linearly polarized wave**; the direction of the $\mathbf{E}$ field is constant, with the angle given by

$$\theta = \tan^{-1}(E_2/E_1)\ . \tag{7.3.8}$$

Elliptical and Circular Polarization

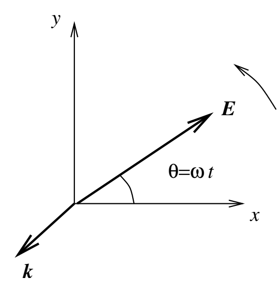
If E_1 and E_2 have **different phases**, we say the wave is **elliptically polarized**. The direction of $\mathbf{E}$ is no longer constant.
A special case is that of **circularly polarized waves**. Here E_1 and E_2 have the same magnitude, but differ by a phase of $\pm\pi/2$. Thus we can write

$$\mathbf{E}(\mathbf{x},t) = E_0(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{7.3.9}$$

where E_0 is real. Without loss of generality, we take $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ in the x and y directions respectively. Thus taking the real (physical) part, we find

$$\begin{aligned} E_x &= E_0\cos(kz-\omega t) = E_0\cos(\omega t - kz) \\ E_y &= \mp E_0\sin(kz-\omega t) = \pm E_0\sin(\omega t - kz). \end{aligned}$$

At fixed z, this is just the equation of a circle.



The different signs correspond to rotating to the *left* or rotating to the *right*; these are more commonly known as **positive** and **negative** helicities.
Since it is possible to use *any* two mutually orthogonal vectors as polarization vectors, it is usually for circularly polarized waves to introduce

$$\boldsymbol{\epsilon}^{\pm} = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \tag{7.3.10}$$

with the properties

$$\boldsymbol{\epsilon}^{\pm *}\cdot\boldsymbol{\epsilon}^{\pm} = 1,\ \boldsymbol{\epsilon}^{\pm *}\cdot\boldsymbol{\epsilon}^{\mp} = 0,\ \boldsymbol{\epsilon}^{\pm *}\cdot\mathbf{n} = 0, \tag{7.3.11}$$

so that a general plane-wave solution is

$$\mathbf{E}(\mathbf{x},t) = (E_+\boldsymbol{\epsilon}^+ + E_-\boldsymbol{\epsilon}^-)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{7.3.12}$$

An important question is, given an electric field $\mathbf{E}(\mathbf{x},t)$, how can one determine its polarization properties; one way of specifying the relative importance of the different components is through the **Stokes Parameters**. This is described in *Jackson 7.2*.

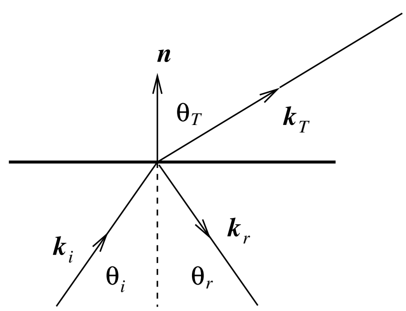
7.4 Reflection and Refraction at Plane Interface between Dielectrics

The laws describing the behavior of a wave at the interface between two media are well known:

1. Angle of reflection = Angle of incidence

2. $\sin\theta_i/\sin\theta_t = n'/n$ (Snells's law) where n', n are the refractive indices of the final and initial media respectively.

These are simple kinematic laws; we would like to determine dynamic properties - intensities and phase changes.



We begin by writing

$$\text{Incident wave:} \quad \begin{aligned} \mathbf{E_i} &= \mathbf{E_0^i}\, e^{i(\mathbf{k_i}\cdot\mathbf{x}-\omega t)} \\ \mathbf{B_i} &= \frac{1}{\omega}\mathbf{k_i}\times\mathbf{E_i} \end{aligned}$$

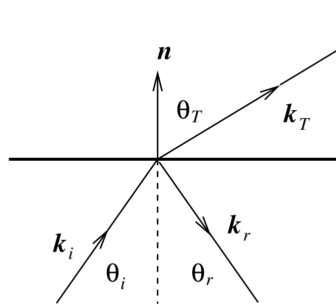
$$\text{Reflected wave:} \quad \begin{aligned} \mathbf{E_r} &= \mathbf{E_0^r}\, e^{i(\mathbf{k_r}\cdot\mathbf{x}-\omega t)} \\ \mathbf{B_r} &= = \frac{1}{\omega}\mathbf{k_r}\times\mathbf{E_r} \end{aligned}$$

$$\text{Refracted wave:} \quad \begin{aligned} \mathbf{E_t} &= \mathbf{E_0^t}\, e^{i(\mathbf{k_t}\cdot\mathbf{x}-\omega t)} \\ \mathbf{B_t} &= \frac{1}{\omega}\mathbf{k_t}\times\mathbf{E_t} \end{aligned}$$

where $k_i^2 = k_r^2 = \mu\epsilon\omega^2$ and $k_t^2 = \mu'\epsilon'\omega^2$.

Boundary Conditions at Interface

We first observe that the boundary conditions must be satisfied $\forall x, y$ at all times t. Thus all fields must have the same phase factor at $z = 0$.
N.B.: We have implicitly assumed this in saying that the frequency in $z > 0$ must be the same as that in $z < 0$.
Thus $\mathbf{k_i} \cdot \mathbf{x} = \mathbf{k_r} \cdot \mathbf{x} = \mathbf{k_t} \cdot \mathbf{x}$ at $z = 0$. This means that the vectors $\mathbf{k_r} - \mathbf{k_i}$ and $\mathbf{k_t} - \mathbf{k_i}$ are orthogonal to any vector $\mathbf{x}$ on the interface plane, i.e. they should be proportional to the unit vector $\mathbf{n}$ normal to this plane. In other words, we should have $\mathbf{k_r} = \mathbf{k_i} + \alpha_r \mathbf{n}$ and $\mathbf{k_t} = \mathbf{k_i} + \alpha_t \mathbf{n}$, where α_r, α_t are some numbers, which means that the vectors $\mathbf{k_i}, \mathbf{k_r}, \mathbf{k_t}$ all lie in the plane formed by $\mathbf{k_i}$ and $\mathbf{n}$ – **plane of incidence**.



From the Figure, we see that

$$
\begin{aligned}
\mathbf{k_i} \cdot \mathbf{x} &= |\mathbf{x}||\mathbf{k_i}| \cos(\pi/2 - \theta_i) = |\mathbf{x}||\mathbf{k_i}| \sin\theta_i \\
\mathbf{k_r} \cdot \mathbf{x} &= |\mathbf{x}||\mathbf{k_r}| \cos(\pi/2 - \theta_r) = |\mathbf{x}||\mathbf{k_r}| \sin\theta_r
\end{aligned}
$$

and since $|\mathbf{k_i}| = |\mathbf{k_r}|$ we have

$$\boxed{\theta_i = \theta_r \qquad (7.4.1)}$$

Similarly,

$$
\begin{aligned}
|\mathbf{k_i}| \sin\theta_i &= |\mathbf{k_t}| \sin\theta_t \\
\Longrightarrow \sqrt{\mu\epsilon} \sin\theta_i &= \sqrt{\mu'\epsilon'} \sin\theta_t.
\end{aligned}
$$

and thus

$$\boxed{\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n} \qquad (7.4.2)}$$

Thus both laws are purely kinematic properties.

Maxwell's equations are

$$
\begin{aligned}
\nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0 \, , \\
\nabla \times \mathbf{H} + i\omega \mathbf{D} &= 0 \\
\nabla \cdot \mathbf{D} &= 0 \, , \\
\nabla \cdot \mathbf{B} &= 0 \, .
\end{aligned} \tag{7.4.3}
$$

Using Gauss's box for divergence equations, and Stokes' contour for the curl equation, we get the boundary conditions at the interface:

$$
\begin{array}{ll}
\mathbf{E}^{\parallel} & \text{is continuous} \\
\mathbf{H}^{\parallel} & \text{is continuous} \\
\mathbf{D}^{\perp} & \text{is continuous} \\
\mathbf{B}^{\perp} & \text{is continuous}
\end{array}
$$

Tangential (to the interface plane) components of fields $\mathbf{E}, \mathbf{H}$ are projected by forming the vector products $\mathbf{E} \times \mathbf{n}$ and $\mathbf{H} \times \mathbf{n}$, while the normal components of $\mathbf{D}$ and $\mathbf{B}$ are given by the scalar products $\mathbf{D} \cdot \mathbf{n}$ and $\mathbf{B} \cdot \mathbf{n}$. Applying to the fields at the interface, we have

$$(\mathbf{E_0^i} + \mathbf{E_0^r} - \mathbf{E_0^t}) \times \mathbf{n} = 0 \tag{7.4.4}$$

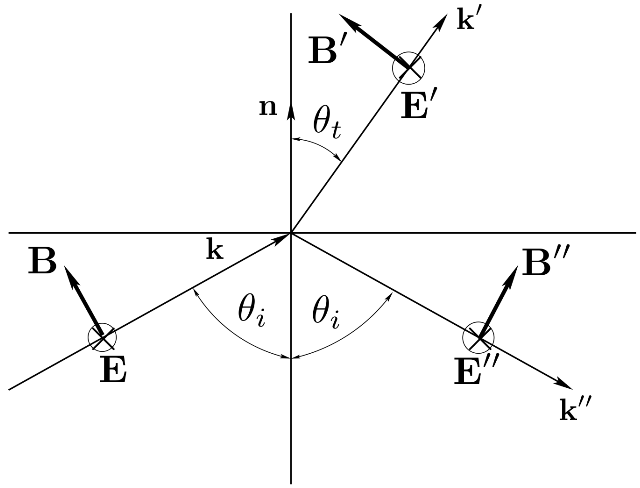
$$\left[\frac{1}{\mu}(\mathbf{k_i} \times \mathbf{E_0^i} + \mathbf{k_r} \times \mathbf{E_0^r}) - \frac{1}{\mu'}\mathbf{k_t} \times \mathbf{E_0^t}\right] \times \mathbf{n} = 0 \tag{7.4.5}$$

$$\left[\epsilon(\mathbf{E_0^i} + \mathbf{E_0^r}) - \epsilon' \mathbf{E_0^t}\right] \cdot \mathbf{n} = 0 \tag{7.4.6}$$

$$\left[\mathbf{k_i} \times \mathbf{E_0^i} + \mathbf{k_r} \times \mathbf{E_0^r} - \mathbf{k_t} \times \mathbf{E_0^t}\right] \cdot \mathbf{n} = 0. \tag{7.4.7}$$

We we now consider two cases; where the electric polarization vector is *normal to plane of incidence* and where it is *parallel to plane of incidence.*

7.4.1 Normal to Plane of Incidence



The z axis is normal to the interface (i.e. $\hat{\mathbf{n}}$ is in z-direction), and we choose the x axis to be in the plane of incidence as shown. Thus the electric field is along the y axis. The first boundary condition Eq. (7.4.4) yields

$$E_0^i + E_0^r = E_0^t \ . \tag{7.4.8}$$

We now turn to the second second boundary condition Eq. (7.4.5). The first term yields

$$\begin{aligned} \frac{1}{\mu}\left[\mathbf{k_i}\times\mathbf{E_0^i}\right]\times\mathbf{n} &= \frac{1}{\mu}\left[\mathbf{E_0^i}(\mathbf{n}\cdot\mathbf{k_i}) - \mathbf{k_i}(\mathbf{n}\cdot\mathbf{E_0^i})\right] \\ &= \frac{1}{\mu}\mathbf{E_0^i}|\mathbf{k_i}|\cos\theta_i = \omega\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i\mathbf{E_0^i} \end{aligned}$$

(we used $(\mathbf{n}\cdot\mathbf{E_0^i}) = 0$). Treating the other two terms similarly, we find

$$(E_0^i - E_0^r)\omega\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \omega\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t E_0^t = 0, \tag{7.4.9}$$

yielding

$$\cos\theta_i\sqrt{\frac{\epsilon}{\mu}}\left(E_0^i - E_0^r\right) - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t E_0^t = 0 \tag{7.4.10}$$

or

$$E_0^i - E_0^r = \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_t}{\cos\theta_i}E_0^t \tag{7.4.11}$$

Since $\mathbf{E_0^i}, \mathbf{E_0^r}$ and $\mathbf{E_0^t}$ for this polarization are transverse to $\mathbf{n}$, one of the remaining boundary conditions (7.4.6) is satisfied trivially, while another one (7.4.7) yields no new information. Indeed,

$$\left[\mathbf{k_i}\times\mathbf{E_0^i} + \mathbf{k_r}\times\mathbf{E_0^r} - \mathbf{k_t}\times\mathbf{E_0^t}\right]\cdot\mathbf{n} = 0. \tag{7.4.12}$$

may be written as

$$(\mathbf{n} \times \mathbf{k_i}) \cdot \mathbf{E_0^i} + (\mathbf{n} \times \mathbf{k_r}) \cdot \mathbf{E_0^r} - (\mathbf{n} \times \mathbf{k_t}) \cdot \mathbf{E_0^t} = 0 \tag{7.4.13}$$

or

$$k_i E_0^i \sin\theta_i + k_r E_0^r \sin\theta_i - k_t E_0^t \sin\theta_t = 0 \tag{7.4.14}$$

and

$$\sqrt{\mu\epsilon}(E_0^i + E_0^r)\sin\theta_i = \sqrt{\mu'\epsilon'} E_0^t \sin\theta_t \Rightarrow E_0^i + E_0^r = E_0^t \ . \tag{7.4.15}$$

On the last step, we used the Snell's law $\sqrt{\epsilon'\mu'}/\sqrt{\epsilon\mu} = \sin\theta_i / \sin\theta_t$.

Thus, adding and subtracting Eqs. (7.4.8) and (7.4.11) we find

$$\boxed{\begin{aligned}
\frac{E_0^r}{E_0^i} &= \frac{1 - \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{1 - \frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_t}}{1 + \frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_t}} && (7.4.16)\\
\frac{E_0^t}{E_0^i} &= \frac{2}{1 + \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{2}{1 + \frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_t}} && (7.4.17)
\end{aligned}}$$

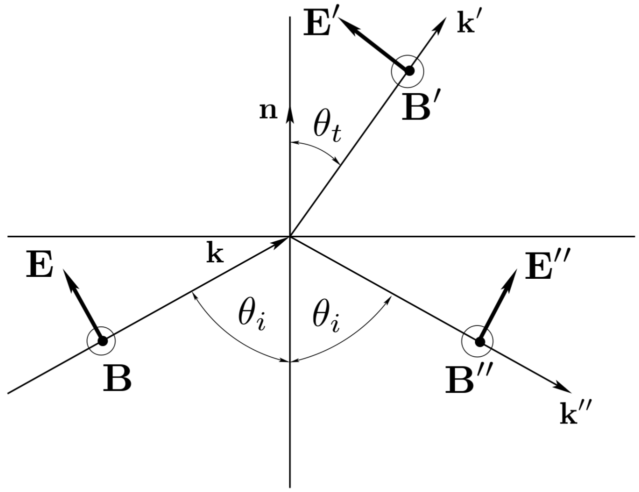
Here we used the Snell's law again.

For visible light, we can usually put $\mu = \mu'$, giving

$$\begin{aligned}
\frac{E_0^r}{E_0^i} &= \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_i + \theta_t)} \\
\frac{E_0^t}{E_0^i} &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t + \theta_i)}.
\end{aligned}$$

This is just **Fresnel's formula** for light polarized perpendicular to plane of incidence.

7.4.2 Electric Field in Plane of Incidence



Here we use 1) boundary condition that $\mathbf{E}^{\parallel}$ is continuous, i.e. Eq. (7.4.4), to yield

$$(E_0^i - E_0^r)\cos\theta_i = E_0^t \cos\theta_t \tag{7.4.18}$$

(it is x-component of $\mathbf{E}$ that is parallel to interface plane, hence $\cos\theta$ factors), and 2) boundary condition that $\mathbf{H}^{\parallel}$ is continuous, i.e. Eq. (7.4.5), to give

$$\sqrt{\frac{\epsilon}{\mu}}\left(E_0^i + E_0^r\right) = \sqrt{\frac{\epsilon'}{\mu'}}E_0^t \tag{7.4.19}$$

(now all $\mathbf{H}$-fields are oriented in y-direction, recall that $\omega\mathbf{H} = \mathbf{k}\times\mathbf{E}/\mu$). Combining

$$E_0^i - E_0^r = E_0^t\,\frac{\cos\theta_t}{\cos\theta_i} \tag{7.4.20}$$

and

$$E_0^i + E_0^r = \sqrt{\frac{\epsilon'\mu}{\mu'\epsilon}}E_0^t \tag{7.4.21}$$

gives

$$\boxed{\begin{aligned}
\frac{E_0^r}{E_0^i} &= \frac{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} - \frac{\cos\theta_t}{\cos\theta_i}}{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} + \frac{\cos\theta_t}{\cos\theta_i}} = \frac{1 - \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{1 - \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t}}{1 + \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t}} && (7.4.22)\\
\frac{E_0^t}{E_0^i} &= \frac{2}{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} + \frac{\cos\theta_t}{\cos\theta_i}} = \frac{2\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}}{1 + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{2\frac{n'}{n}\frac{\epsilon}{\epsilon'}}{1 + \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t}} && (7.4.23)
\end{aligned}}$$

If $\mu = \mu'$, then $\epsilon/\epsilon' = \sin^2\theta_t/\sin^2\theta_i = n^2/n'^2$, and we have

$$\frac{E_0^r}{E_0^i} = \frac{1 - \frac{\sin^2\theta_t}{\sin^2\theta_i}\frac{\tan\theta_i}{\tan\theta_t}}{1 + \frac{\sin^2\theta_t}{\sin^2\theta_i}\frac{\tan\theta_i}{\tan\theta_t}} = \frac{1 - \frac{\sin\theta_t}{\sin\theta_i}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \frac{\sin\theta_t}{\sin\theta_i}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{7.4.24}$$

and

$$\frac{E_0^t}{E_0^i} = \frac{2\frac{\sin\theta_t}{\sin\theta_i}}{1 + \frac{\sin^2\theta_t}{\sin^2\theta_i}\frac{\tan\theta_i}{\tan\theta_t}} = \frac{2\frac{\sin\theta_t}{\sin\theta_i}}{1 + \frac{\sin\theta_t}{\sin\theta_i}\frac{\cos\theta_t}{\cos\theta_i}} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_i + \sin 2\theta_t} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \tag{7.4.25}$$

(here we used $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$ and $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$).

Incident Wave Normal to Interface

Consider the case $\theta_i = 0$. A subtle point is that the **k** and **n** are parallel, so there is no unique plane of incidence. Taking θ_i small but finite and assuming that the wave is polarized *perpendicular* to the plane of incidence, in the limit $\theta_i \to 0$ we find from Eq. (7.4.16)

$$\frac{E_0^r}{E_0^i} = \frac{1 - \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}}{1 + \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}} = \frac{1 - \frac{\mu}{\mu'}\frac{n'}{n}}{1 + \frac{\mu}{\mu'}\frac{n'}{n}} \longrightarrow \frac{n - n'}{n + n'} \quad \text{if } \mu = \mu' \tag{7.4.26}$$

$$\frac{E_0^t}{E_0^i} = \frac{2}{1 + \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}} = \frac{2}{1 + \frac{\mu}{\mu'}\frac{n'}{n}} \longrightarrow \frac{2}{1 + n'/n} \quad \text{if } \mu = \mu' \tag{7.4.27}$$

Note, that according to the figure that we used in this case, the positive values of both E_0^i and E_0^r correspond to positive y-direction.
Now, taking again θ_i small, but finite and assuming that the wave is polarized *parallel* to the plane of incidence, in the limit $\theta_i \to 0$ we find from Eq. (7.4.22)

$$\frac{E_0^r}{E_0^i} = \frac{1 - \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}}{1 + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}} = \frac{1 - \frac{\mu'}{\mu}\frac{n}{n'}}{1 + \frac{\mu'}{\mu}\frac{n}{n'}} \longrightarrow \frac{n' - n}{n + n'} \quad \text{if } \mu = \mu' ,$$

i.e., this ratio has an apparently opposite sign. Note, however, that according to the figure that we used in this case, the positive value of E_0^i corresponds to the negative x-direction, while positive value of E_0^r corresponds to the positive x-direction.
For another ratio, we find

$$\frac{E_0^t}{E_0^i} = \frac{2\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}}{1 + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}} = \frac{2}{1 + \sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}} ,$$

the same result as in Eq. (7.4.27).
Thus we see that, if both refractive indices are equal

$$\begin{aligned} E_0^r &= 0 \\ E_0^t &= E_0^r \end{aligned}$$

as expected. If the second media is a conductor, $n' \longrightarrow \infty$, then all of the wave is reflected, with

$$E_0^r = E_0^i \tag{7.4.28}$$

(the sign convention is assumed that was used for the case of polarization in the plane of incidence).

7.5 Brewster's Angle and Total Internal Reflection

7.5.1 Brewster's Angle

In the case of polarization in the plane of incidence, we have

$$\frac{E_0^r}{E_0^i} = \frac{1 - \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t}}{1 + \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t}}. \tag{7.5.1}$$

There is an angle for which *no* wave is reflected, given by

$$\frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_t} = 1. \tag{7.5.2}$$

Setting $\mu = \mu' = 1$, we can also use Eq. (7.4.24) that gives

$$\frac{E_0^r}{E_0^i} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{\sin(\theta_i - \theta_t)\cos(\theta_i + \theta_t)}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \ . \tag{7.5.3}$$

We find that the ratio E_0^r/E_0^i vanishes if

$$\sin(\theta_i - \theta_t)\cos(\theta_i + \theta_t) = 0 \ . \tag{7.5.4}$$

One of the solutions of this equation, namely, $\theta_i = \theta_t$ holds only if $n' = n$. This corresponds to situation, when optical properties of two substances are the same, and naturally there is no reflection on their interface. Another solution is

$$\theta_i + \theta_t = \pi/2 \ , \tag{7.5.5}$$

in which case $\sin\theta_t = \cos\theta_i$. Substituting this result in Snell's law

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n'}{n} \ , \tag{7.5.6}$$

we have

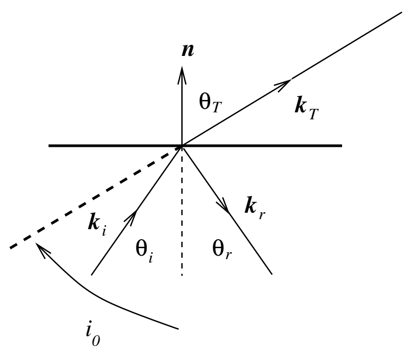
$$\tan\theta_i = \frac{n'}{n}, \tag{7.5.7}$$

or

$$\theta_i = \tan^{-1}\left(\frac{n'}{n}\right). \tag{7.5.8}$$

This is **Brewster's Angle**. If we have a plane wave of mixed polarization incident at this angle, the reflected radiation only has a polarization component perpendicular to the plane of incidence. It is a simple way to produce plane-polarized light.

7.5.2 Total Internal Reflection



According to Snell's law,

$$\sin\theta_t = (n/n')\sin\theta_i \ .$$

Thus, if light passes from a medium of higher optical density to one of lower optical density, the **angle of refraction** is **greater** than the **angle of incidence**.
Hence there is an angle of incidence i_0 for which $\theta_t = \pi/2$, given by

$$\sin i_0 = n'/n \ . \tag{7.5.9}$$

From Snell's law, we have in general

$$\begin{aligned} \cos\theta_t = \sqrt{1-\sin^2\theta_t} &= \sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_i} \\ &= \sqrt{1-\left(\frac{\sin\theta_i}{\sin i_0}\right)^2}. \end{aligned}$$

For $\theta_i > i_0$, $\cos\theta_t$ becomes purely imaginary. Thus the refractive wave has a phase factor

$$\begin{aligned} e^{i\mathbf{k_t}\cdot\mathbf{x}} &= e^{i(k_t x\sin\theta_t + k_t z\cos\theta_t)} \\ &= e^{ik_t x(n/n')\sin\theta_i} e^{-k_t z\sqrt{(\sin\theta_i/\sin i_0)^2-1}}. \end{aligned}$$

We see that the refracted wave propagates **parallel to the surface**, and is **exponentially attenuated** with increasing z. The attenuation occurs over only a few wavelengths *unless* $\theta_i \approx i_0$.

Note that the time-averaged energy flux across the interface is

$$\langle \mathbf{S}\cdot\mathbf{n}\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{n}\cdot(\mathbf{E_t}\times\mathbf{H_t^*})\right]. \tag{7.5.10}$$

Now $\mathbf{H_t} = (\mathbf{k_t}\times\mathbf{E_t})/\mu'\omega$, and thus

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{E_t}\times\mathbf{H_t^*}) &= \mathbf{n}\cdot\left[\mathbf{E_t}\times(\mathbf{k_t}\times\mathbf{E_t^*})\right]/\mu'\omega \\ &= |\mathbf{E_t}|^2\mathbf{n}\cdot\mathbf{k_t}/\mu'\omega, \end{aligned}$$

whence

$$\begin{aligned} \langle \mathbf{S}\cdot\mathbf{n}\rangle &= \frac{1}{2}\mathrm{Re}\left[|\mathbf{E_t}|^2\mathbf{n}\cdot\mathbf{k_t}\right]/\mu'\omega \\ &= \frac{1}{2}\mathrm{Re}\left[|\mathbf{E_t}|^2 k_t\cos\theta_t\right]/\mu'\omega \\ &= 0, \end{aligned}$$

since $\cos\theta_t$ is purely imaginary; there is no *time-averaged* energy flux across the interface. The principle of total internal reflection has many applications, most notably in fibre-optic cables. The analysis presented here assumes, of course, that the material is wide compared to the wave length of light.

7.6 Dispersion

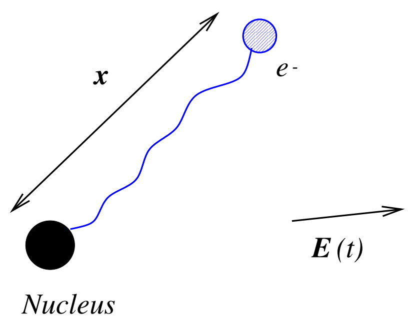
So far, we have been investigating the propagation of waves of a fixed frequency. The wave number is related by

$$k^2 = \mu\epsilon\omega^2. \tag{7.6.1}$$

Suppose now we consider a wave having a spread of frequencies. In general, the values of μ and, in particular, ϵ are frequency dependent, and thus different frequencies have different propagation properties. This is called **dispersion**.

7.6.1 Simple Model for Dispersion

Consider an electron of mass m and charge $-e$, bound to a (fixed) nucleus by a harmonic potential with resonant frequency ω_0, and a damping term with damping constant γ. In the absence of an external electric field, the electron will undergo damped simple-harmonic motion about an equlibrium.



We now apply an external electromagnetic field $(\mathbf{E}, \mathbf{B})$. Then the force on the electron is

$$\mathbf{F}(t) = -e(\mathbf{E}(t) + \mathbf{v} \times \mathbf{B}(t)). \tag{7.6.2}$$

Providing the velocity is small compared to that of light, the magnetic force will be negligible; recall that $c|\mathbf{B}| \approx |\mathbf{E}|$. Thus the equation of motion of the electron is

$$m\left(\frac{d^2}{dt^2}\mathbf{x} + \gamma\frac{d}{dt}\mathbf{x} + \omega_0^2\mathbf{x}\right) = -e\mathbf{E}(t). \tag{7.6.3}$$

The dipole moment of the system is given by $\mathbf{p} = -e\mathbf{x}$. We now assume that the external field has frequency ω, so that the time dependence is

$$\mathbf{E} = \mathbf{E_0}e^{-i\omega t}. \tag{7.6.4}$$

Thus the displacement will have the same frequency dependence, and we have an equation of motion

$$m(-\omega^2 - i\omega\gamma + \omega_0^2)\mathbf{x} = -e\mathbf{E_0}, \tag{7.6.5}$$

yielding a dipole moment

$$\mathbf{p} = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}\,\mathbf{E_0}\ . \tag{7.6.6}$$

We now consider the case of N atoms/unit volume, each having Z electrons of which f_j electrons have resonant frequency ω_j. We will take this as a model for a linear medium, in which the polarization $\mathbf{P}$ arises solely from the applied external field. Thus, recalling that

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \tag{7.6.7}$$

and $\epsilon = \epsilon_0(1+\chi_e)$, we find

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \ . \tag{7.6.8}$$

with $\sum_j f_j = Z$. We can rewrite this expression as

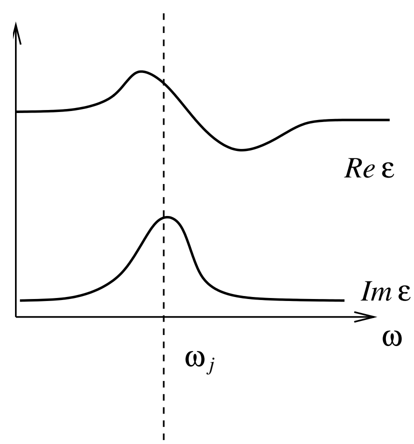
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j f_j \frac{(\omega_j^2 - \omega^2) + i\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}. \tag{7.6.9}$$

We have thus seen how even a simple model gives a frequency-dependent permittivity.

7.6.2 Permittivity in Resonance Region

In general, we can assume that the damping factor γ is small. From the form of Eq. (7.6.9), it is clear that at very low frequencies, the susceptibility is positive and the relative permittivity greater than one. As successive resonant frequencies are passed, more negative terms contribute and eventually the relative permittivity is less than one.

Particularly interesting is the behavior in the neighborhood of a resonance.



Here the real part of $\epsilon(\omega)$ is peaked around ω_j, and furthermore displays *anomalous dispersion* in which light of higher frequency is less refracted than light of lower frequency.

The presence of an appreciable imaginary part of $\epsilon(\omega)$ near $\omega = \omega_j$ represents **absorption**; energy dissipated in the medium. To see how this arises, consider a wave propagating in the z-direction. We will write the wave number as

$$k = \beta + i\alpha/2; \qquad \text{amplitude} \approx e^{-\alpha z/2}. \tag{7.6.10}$$

Thus α clearly represents absorption of the wave. Setting $\mu = \mu_0$, and recalling $k = \sqrt{\mu\epsilon}\omega$, we have

$$(\beta^2 - \alpha^2/4) + i\alpha\beta = (\sqrt{\mu_0\epsilon_0})^2\omega^2\epsilon/\epsilon_0 \tag{7.6.11}$$

which gives

$$\left.\begin{array}{rcl} \beta^2 - \alpha^2/4 & = & \frac{\omega^2}{c^2}\mathrm{Re}\,\epsilon/\epsilon_0 \\ \alpha\beta & = & \frac{\omega^2}{c^2}\mathrm{Im}\,\epsilon/\epsilon_0 \end{array}\right\} . \tag{7.6.12}$$

Note that if $\alpha \ll \beta$, we have

$$\begin{array}{rcl} \alpha & = & \dfrac{\mathrm{Im}\,\epsilon(\omega)}{\mathrm{Re}\,\epsilon(\omega)}\beta \ , \\ \text{where} & & \\ \beta & = & \dfrac{\omega}{c}\sqrt{\mathrm{Re}\,\epsilon/\epsilon_0} \ . \end{array}$$

7.6.3 Low Frequency behavior and Electrical Conductivity

In a conductor, some of the electrons can move freely. Thus there are some electrons with resonant frequency $\omega_0 = 0$, whose contribution to the permittivity is

$$\epsilon(\omega) = \tilde{\epsilon}(\omega) + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \ , \tag{7.6.13}$$

where $\tilde{\epsilon}$ represents the background permittivity coming from all the other modes. We see from this that $\epsilon(\omega)$ is singular as $\omega \longrightarrow 0$, and we will now relate this property to electrical conductivity.

Our starting point is the Maxwell-Ampère law (ME3):

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{7.6.14}$$

We will now impose that $\mathbf{J}$ and $\mathbf{E}$ are related through Ohm's law

$$\mathbf{J} = \sigma\mathbf{E} \ , \tag{7.6.15}$$

where σ is the conductivity. If we assume the usual frequency behavior $\exp(-i\omega t)$, and assume the background permittivity is a constant $\tilde{\epsilon}(\omega) = \epsilon_b$, Eq. (7.6.14) becomes

$$\nabla \times \mathbf{H} = -i\omega\left[\epsilon_b + i\frac{\sigma}{\omega}\right]\mathbf{E}. \tag{7.6.16}$$

An alternative procedure is to ascribe all properties, **including current flow**, to the dielectric properties of the medium. In that case we have

$$\nabla \times \mathbf{H} = -i\omega\mathbf{D} = -i\omega\left[\epsilon_b + i\frac{Ne^2 f_0}{m\omega(\gamma - i\omega)}\right]\mathbf{E}. \tag{7.6.17}$$

Comparing Eqs. (7.6.16) and (7.6.17), we find

$$i\,\frac{\sigma}{\omega} = i\,\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\ , \tag{7.6.18}$$

i.e.

$$\boxed{\sigma = \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)}\ . \qquad (7.6.19)}$$

Note that we can rewrite this expression as

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}\ , \tag{7.6.20}$$

where

$$\sigma_0 = \frac{N f_0 e^2}{m\gamma_0}, \tag{7.6.21}$$

and $\tau = \gamma_0^{-1}$. Essentially, we have

- $N f_0$ is *number of free electrons per unit volume.*
- γ_0/f_0 is *damping constant*, determined experimentally.

For good conductors $\gamma_0/f_0 \simeq 4 \times 10^{13}\,\mathrm{s}^{-1}$. If we assume $f_0 \simeq 1$, then $\omega\tau$ is small till the microwave region $\omega \simeq 10^{11}\,\mathrm{s}^{-1}$; σ **is real**.
Note that if $\omega/\gamma_0 \gg 1$, then σ is purely imaginary, and we have a phase shift between **E** and **J**.

7.7 High-Frequency behavior and Plasma Frequency

Suppose that ω is much larger than the highest resonance frequency. Then we have

$$\begin{aligned}\frac{\epsilon}{\epsilon_0} &= 1+\frac{Ne^2}{\epsilon_0 m}\sum_j f_j\frac{(\omega_j^2-\omega^2)+i\omega\gamma_j}{(\omega_j^2-\omega^2)^2+\omega^2\gamma_j^2} \\ &\xrightarrow{\omega/\omega_j\gg 1} 1-\frac{Ne^2}{\epsilon_0 m}\sum_j f_j\frac{\omega^2}{\omega^4} \\ &= 1-\omega_P^2/\omega^2\ ,\end{aligned} \tag{7.7.1}$$

where

$$\omega_P^2=\frac{NZe^2}{\epsilon_0 m} \tag{7.7.2}$$

is the $\boxed{\textbf{plasma frequency}}$, so called because all the electrons essentially behave as if free. Recalling that

$$k=\sqrt{\mu\epsilon}\,\omega|_{\mu=\mu_0}=\frac{1}{c}\sqrt{\frac{\epsilon}{\epsilon_0}}\omega\ , \tag{7.7.3}$$

where c is the velocity of light *in vacuum*, we have

$$ck=\sqrt{\omega^2-\omega_P^2} \tag{7.7.4}$$

whence

$$\omega^2(k)=\omega_P^2+c^2k^2. \tag{7.7.5}$$

Such an expression, describing the relationship between wave number and frequency, is known as a $\boxed{\textbf{dispersion relation}}$. Similar expressions occur in many places in physics, including special relativity and sound propagation.

In a typical dielectric, when $\omega^2 \gg \omega_P^2$, the dielectric constant is slightly less than, but close to, unity.

In a true plasma, such as the ionosphere, all the electrons are essentially free, and the expression Eq. (7.7.1) is valid for a range of frequencies, *including* $\omega < \omega_P$. The wave number k is purely imaginary for frequencies less than the plasma frequency. Thus a wave incident on a plasma is attenuated in the direction of propagation, with intensity

$$I\propto e^{-2\sqrt{\omega_P^2-\omega^2}z/c}\xrightarrow{\omega\to 0} e^{-2\omega_P z/c}. \tag{7.7.6}$$

7.7.1 Model of Wave Propagation in the Atmosphere

The above plasma model for the ionosphere is modified considerably through the presence of the earth's magnetic field. In the model we now construct, we assume propagation parallel to the earth's field $\mathbf{B_0}$. We assume that there is a force acting on the charges due to a propagating *electric* field, but that the only magnetic force is that arising from the earth's field; recall once again that $c|\mathbf{B}| \simeq |\mathbf{E}|$.
Thus the equation of motion for an electron of charge $-e$ and mass m is

$$m\frac{d^2\mathbf{x}}{dt^2} = -e\mathbf{v} \times \mathbf{B_0} - e\mathbf{E}. \tag{7.7.7}$$

Once again, we consider a monochromatic plane wave with time dependence

$$e^{-i\omega t}. \tag{7.7.8}$$

It is convenient to consider the case of *circularly polarized* waves, for which we introduce the complex polarization vectors

$$\begin{aligned} \boldsymbol{\epsilon}_\pm &= \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon_1} \pm i\boldsymbol{\epsilon_2}) \\ \boldsymbol{\epsilon_3} &= \hat{\mathbf{k}} \qquad \text{(Normal in direction of } \mathbf{k}). \end{aligned}$$

Thus we have

$$\mathbf{x} = x_+\boldsymbol{\epsilon}_+ + x_-\boldsymbol{\epsilon}_- + x_3\boldsymbol{\epsilon_3}, \tag{7.7.9}$$

so that the equation of motion becomes

$$\begin{aligned} m\left[\frac{d^2x_+}{dt^2}\boldsymbol{\epsilon}_+ + \frac{d^2x_-}{dt^2}\boldsymbol{\epsilon}_- + \frac{d^2x_3}{dt^2}\boldsymbol{\epsilon_3}\right] -eB_0\boldsymbol{\epsilon_3} \times \left[\frac{dx_+}{dt}\boldsymbol{\epsilon}_+ + \frac{dx_-}{dt}\boldsymbol{\epsilon}_- + \frac{dx_3}{dt}\boldsymbol{\epsilon_3}\right] \\ = -e\left[E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-\right]e^{-i\omega t}. \end{aligned} \tag{7.7.10}$$

First, it is easy to see that since $\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon_3} = 0$, the motion along the Z direction is free: $x_3 = x_{30} + v_3 t$. Since the forces acting in XY plane are periodic, the motion of the charges in the XY plane will be periodic, too: $x_+(t) = x_+e^{-i\omega t}, \quad x_-(t) = x_-e^{-i\omega t}$.
Now

$$\begin{aligned} \boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_+ &= \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon_2}) = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon_2} - i\boldsymbol{\epsilon_1}) = \\ &= -\frac{i}{\sqrt{2}}(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2), \end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_- &= \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_1 - i\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon_2}) = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon_2} + i\boldsymbol{\epsilon_1}) = \\ &= \frac{i}{\sqrt{2}}(\boldsymbol{\epsilon}_1 - i\boldsymbol{\epsilon}_2),\end{aligned}$$

yielding

$$\begin{aligned}\boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_+ &= -i\boldsymbol{\epsilon}_+ \\ \boldsymbol{\epsilon_3} \times \boldsymbol{\epsilon}_- &= i\boldsymbol{\epsilon}_-\end{aligned} \tag{7.7.11}$$

and thus

$$-\omega^2 m[x_+\boldsymbol{\epsilon}_+ + x_-\boldsymbol{\epsilon}_-] + i\omega e B_0[-ix_+\boldsymbol{\epsilon}_+ + ix_-\boldsymbol{\epsilon}_-] = -e[E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-]. \tag{7.7.12}$$

Looking at the individual components, we find

$$\begin{aligned}-\omega^2 m x_+ + \omega e B_0 x_+ &= -eE_+ \\ -\omega^2 m x_- - \omega e B_0 x_- &= -eE_-\end{aligned}$$

which we may write

$$x_\pm = \frac{e}{m\omega(\omega \mp eB_0/m)}E_\pm = \frac{e}{m\omega(\omega \mp \omega_B)}E_\pm, \tag{7.7.13}$$

where we have introduced

$$\omega_B \equiv eB_0/m, \tag{7.7.14}$$

the frequency of precession of a charged particle in a magnetic field.
Note, that in the absence of the electromagnetic wave, we will have

$$\begin{aligned}-\omega^2 x_+ + \omega\omega_B x_+ &= 0\ , \\ \omega^2 x_- + \omega\omega_B x_- &= 0\ .\end{aligned}$$

For x_+, we have a nontrivial solution $\omega = \omega_B$ that results in $e^{-i\omega_B t}$ time dependence of $x_+(t)$. In case of x_-, we have a trivial solution $x_- = 0$ only. Thus, $\mathbf{x}_\perp(t) = \boldsymbol{\epsilon}_+ e^{-i\omega_B t} x_+^0$, which corresponds to a counterclockwise rotation of electron around the z-axis, i.e. around the magnetic field $\mathbf{B}_0$.
Now, recalling that $\mathbf{p} = -e\mathbf{x}$, we have a dipole moment of the particle

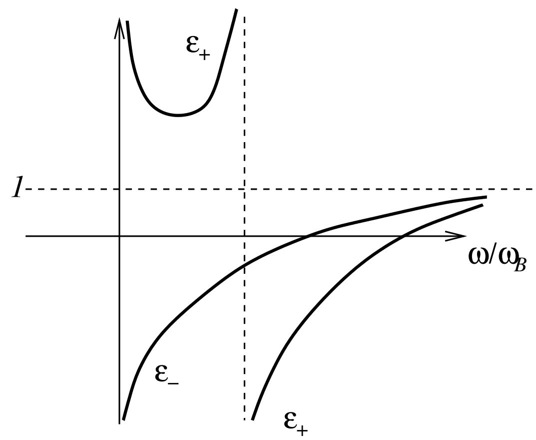
$$p_\pm = \frac{-e^2}{m\omega(\omega \mp \omega_B)}E_\pm. \tag{7.7.15}$$

Thus, using the expression for the plasma frequency Eq. (7.7.2), the polarization may be written

$$P_{\pm} = -\epsilon_0 \frac{\omega_P^2}{\omega(\omega \mp \omega_B)} E_{\pm} \tag{7.7.16}$$

whence

$$\boxed{\epsilon_{\pm}/\epsilon_0 = 1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_B)} \qquad (7.7.17)}$$



Thus, in this highly simplified model, we see that the permittivity depends on the polarization of the incident wave. Indeed, for certain ranges of ω we find that the permittivity can be negative, and hence one or both polarizations no longer propagate.

Note also a resonance-type behavior in the vicinity of $\omega = \omega_B$ that occurs for ϵ_+, i.e. for EM waves whose circular polarization coincides with the direction of rotation of an electron in the earth's magnetic field.

7.8 Superposition of Waves and Group Velocity

So far we have considered monochromatic waves, but have seen that, if the medium is dispersive, different frequencies will travel with different velocities. In the section, we will describe how, for a general plane wave, the rate of energy flow is in general different from the phase velocity, or velocity of propagation of a particular frequency component. To simplify the discussion, we will consider the problem in one dimension.

We will write a general wave in terms of its physical components. The dispersive properties are encompassed in the dispersion relation

$$\omega \equiv \omega(k) \tag{7.8.1}$$

where $\omega(-k) = \omega(k)$. The general solution is then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t}, \tag{7.8.2}$$

where the amplitudes $A(k)$ are given by

$$A(k) = \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx \ . \tag{7.8.3}$$

For a monochromatic wave, of wave number k_0, we have

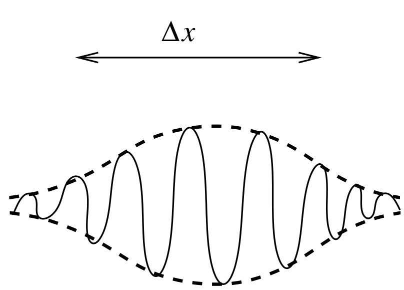
$$u(x,0) = e^{ik_0 x} \tag{7.8.4}$$

yielding

$$A(k) = 2\pi\delta(k - k_0). \tag{7.8.5}$$

In practice we virtually never deal with pure monochromatic plane waves of fixed frequency k_0, but rather with pulses, centered about a frequency k_0. In particular, we will consider the propagation of a Gaussian wave packet, of width Δx, centered at $x = 0$. Then

$$u(x,0) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-x^2/4\Delta x^2} e^{ik_0 x}. \tag{7.8.6}$$



This satisfies

$$\begin{aligned}\int dx\,|u(x,0)|^2 &= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2}\int dx\,e^{-x^2/2\Delta x^2}\\ &= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2}\sqrt{\frac{\pi}{1/2\Delta x^2}} = 1\end{aligned}$$

and

$$\begin{aligned}\langle x^2\rangle &= \int dx\,|u(x,0)|^2 x^2 = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2}\int dx e^{-x^2/2\Delta x^2}x^2\\ &= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2}(-2)\frac{d}{d(1/\Delta x^2)}\int dx\,e^{-x^2/2\Delta x^2} = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2}(-2)\frac{d}{d(1/\Delta x^2)}\sqrt{\frac{\pi}{1/2\Delta x^2}}\\ &= (-2)\left(-\frac{1}{2}\right)\frac{1}{1/\Delta x^2} = \Delta x^2\ ,\end{aligned}$$

showing that the width is indeed Δx.
The amplitudes of the various components are given by

$$\begin{aligned}A(k) = \int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx &\sim \int dx\,e^{-x^2/(2\Delta x)^2}e^{i(k_0-k)x}\\ &\sim e^{-(k_0-k)^2/4(1/2\Delta x)^2} \equiv e^{-(k_0-k)^2/4\Delta k^2}\ .\end{aligned}$$

By analogy with the width of the wave packet, we see that the amplitude $A(k)$ is centered at $k = k_0$, with width

$$\Delta k = \frac{1}{2\Delta x}. \tag{7.8.7}$$

In fact, more generally we have

$$\Delta x \Delta k \geq 1/2\ . \tag{7.8.8}$$

Thus we have the important observation that a short pulse, even of "fixed" frequency k_0, contains a spread of monochromatic components. This expression, of course, is more familiar from *Heisenberg's Uncertainty Principle.*

7.8.1 Group Velocity

To see how this spread of frequencies effects the propagation of a wave, we consider the simple case of two monochromatic waves, of the same amplitude and of neighboring frequencies (k_1, ω_1) and (k_2, ω_2), where $k_1, k_2 \sim k_0$. Then the resulting "wave packet" propagates as

$$\begin{aligned}U(x,t) &= A\left[e^{i(k_1x-\omega_1t)} + e^{i(k_2x-\omega_2t)}\right]\\ &= Ae^{i[(k_1+k_2)x/2-(\omega_1+\omega_2)t/2]}\left\{e^{i[(k_1-k_2)x/2-(\omega_1-\omega_2)t/2]} + e^{i[(k_2-k_1)x/2-(\omega_2-\omega_1)t/2]}\right\}\\ &= 2A\cos\left[\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right]e^{i[(k_1+k_2)x/2-(\omega_1+\omega_2)t/2]}\end{aligned}$$

We have written the wave as a *slowly oscillating amplitude factor* with velocity

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} \longrightarrow \left.\frac{d\omega}{dk}\right|_{k_0} \quad \text{as } k_2 \to k_1, \tag{7.8.9}$$

known as the **group velocity**, and a rapidly oscillating "phase" with velocity

$$v_p = \frac{\omega_1 + \omega_2}{k_1 + k_2} \longrightarrow \frac{\omega}{k} \quad \text{as } k_2 \to k_1 = k. \tag{7.8.10}$$

Since the energy density is associated with the amplitude of the wave, we see that, in this approximation, energy is transmitted with the group velocity, given by Eq. (7.8.9) with k_0 the central value of the wave number.

We now recall the relationship between ω and k

$$\omega(k) = \frac{ck}{n(k)}, \tag{7.8.11}$$

where $n(k)$ is the index of refraction, and c is the velocity of light in a vacuum. The phase velocity can then be written

$$v_p = \frac{\omega(k)}{k} = \frac{c}{n(k)}. \tag{7.8.12}$$

This can be either **less than** or **greater than** the speed of light; for most media at optical frequencies, $n(k) > 1$. We can rewrite the group velocity using Eq. (7.8.11), regarding $k = k(\omega)$:

$$ck(\omega) = \omega n(\omega)$$

and find

$$\begin{aligned} c\frac{dk}{d\omega} &= n(\omega) + \omega\frac{dn}{d\omega} \\ \Longrightarrow v_g = \left.\frac{d\omega}{dk}\right|_{k_0} &= \frac{c}{n(\omega) + \omega dn/d\omega}. \end{aligned}$$

Providing $dn/d\omega > 0$, we have $v_g < v_p$. However, if $dn/d\omega < 0$ (anomalous refraction), v_g can be greater than v_p.

7.8.2 Propagation of a Gaussian wave packet in the dispersive medium

As we discussed, the general solution of the wave equation is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t} \ , \tag{7.8.13}$$

where the amplitudes $A(k)$ can be found from the initial shape $u(x,0) \equiv u_0(x)$ at $t = 0$:

$$A(k) = \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx \ . \tag{7.8.14}$$

If $\omega(k)$ is just a linear function of k, i.e. $\omega = vk$, with v being the phase velocity (equal in this case to the group velocity), then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - ivkt} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ik(x-vt)} = u_0(x - vt) \ , \tag{7.8.15}$$

i.e., the wave packet moves as a whole with velocity v. Assume now that at the initial moment of time $t = 0$ we have a Gaussian pulse

$$u(x,0) \ = \ \left(\frac{1}{\pi L^2}\right)^{1/4} \exp\left\{ -\frac{x^2}{2L^2} + ik_0 x \right\} \tag{7.8.16}$$

normalized by

$$\int_{\infty}^{\infty} |u(x,0)|^2 \, dx = 1 \ . \tag{7.8.17}$$

The parameter L here is related by $L = \sqrt{2}\Delta x$ to the width Δx of the Gaussian wave packet introduced earlier. As shown above, in a linear medium the pulse propagates as a whole with velocity $v = c/n \equiv \omega_0/k_0$:

$$u(x,t) \ = \ \left(\frac{1}{\pi L^2}\right)^{1/4} \exp\left\{ -\frac{(x-vt)^2}{2L^2} + ik_0(x - vt) \right\} . \tag{7.8.18}$$

Suppose at $t = 0$ we switch on the dispersion so that $\omega = \omega(k)$ (some non-linear function). What will happen with the pulse? To this end, we will use the general solution given above

$$u(x,t) \ = \ \int \frac{dk}{2\pi} \ A(k) \ e^{-i\omega(k)t + ikx}$$

with the function $A(k)$ extracted from the initial condition $u(x,0)$ corresponding to Gaussian pulse (7.8.18) at $t=0$. This gives

$$\begin{aligned}
A(k) \;&=\; \int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx = \int_{-\infty}^{\infty} \left(\frac{1}{\pi L^2}\right)^{1/4} \exp\left\{-\frac{x^2}{2L^2}+ik_0x\right\}e^{-ikx}\,dx \\
&= \left(\frac{1}{\pi L^2}\right)^{1/4}\int_{-\infty}^{\infty} \exp\left\{-\frac{[x-i(k_0-k)L^2]^2}{2L^2}-\frac{1}{2}(k-k_0)^2L^2\right\}dx \\
&= \underbrace{\left(\frac{1}{\pi L^2}\right)^{1/4}\sqrt{2\pi L^2}}_{(4\pi L^2)^{1/4}}\exp\left\{-\frac{1}{2}(k-k_0)^2L^2\right\} \,. \qquad (7.8.19)
\end{aligned}$$

A typical behavior of $\omega(k)$ is given by Eq. (7.7.5): $\omega^2=\omega_P^2+c^2k^2$. For simplicity, we will consider an approximate model of the behavior of frequency in the vicinity of ω_0 in the form

$$\omega(k) \;=\; \omega_0\left(1+\frac{a^2k^2}{2}\right) \,, \qquad (7.8.20)$$

where ω_0 is a constant, so that the frequency $\omega(k_0)$ corresponding to the center k_0 of the k-space Gaussian wave packet is given by

$$\omega(k_0) \;=\; \omega_0\left(1+\frac{a^2k_0^2}{2}\right) \,.$$

Substituting these expressions for $A(k)$ and $\omega(k)$, we obtain

$$u(x,t)=\left(4\pi L^2\right)^{1/4}\int\frac{dk}{2\pi}\,\exp\left\{-\frac{1}{2}(k-k_0)^2L^2\right\}\,\exp\left\{-i\omega_0t\left(1+\frac{a^2k^2}{2}\right)+ikx\right\} \,.$$

Writing $k=k_0+k'$, we get $kx=k_0x+k'x$ and $k^2=k_0^2+2k_0k'+(k')^2$, thus

$$\begin{aligned}
u(x,t) =&\left(4\pi L^2\right)^{1/4}\exp\left\{-i\omega_0t\left(1+\frac{a^2k_0^2}{2}\right)+ik_0x\right\} \\
&\times\int\frac{dk}{2\pi}\,\exp\left\{-\frac{1}{2}(k')^2(L^2+i\omega_0a^2t)\right\}\,e^{ik'(x-\omega_0a^2k_0t)} \,. \qquad (7.8.21)
\end{aligned}$$

Taking the Gaussian integral over k' gives

$$u(x,t)=\frac{(4\pi L^2)^{1/4}/\sqrt{2\pi}}{\sqrt{L^2+i\omega_0a^2t}}e^{-i\omega_0t\left(1+\frac{a^2k_0^2}{2}\right)+ik_0x}\,\exp\left\{-\frac{(x-\omega_0a^2k_0t)^2}{2L^2(1+i\omega_0\frac{a^2t}{L^2})}\right\} . \qquad (7.8.22)$$

Note that $\omega_0(1+\frac{a^2k_0^2}{2})=\omega(k_0)$. The peak of the pulse (7.8.22) is located at $x=\omega_0a^2k_0t \Rightarrow$ it moves with the *group velocity* $v_g=\omega_0a^2k_0=\frac{\partial\omega_k}{\partial k}\Big|_{k=k_0}$. Thus, we may write

$$u(x,t)=\left(\frac{1}{\pi L^2}\right)^{1/4}\frac{e^{-i\omega(k_0)t+ik_0x}}{\sqrt{1+iv_gt/k_0L^2}}\,\exp\left\{-\frac{(x-v_gt)^2}{2(L^2+iv_gt/k_0)}\right\} . \qquad (7.8.23)$$

Note also that the phase factor here may be written as $e^{ik_0(x-v_pt)}$, where $v_p \equiv \omega(k_0)/k_0$ is the phase velocity. Writing the modulating wave as

$$\exp\left\{-\frac{(x-v_gt)^2}{2(L^2+iv_gt/k_0)}\right\} = \exp\left\{-\frac{(x-v_gt)^2\,[L^2-iv_gt/k_0]}{2[L^4+(v_gt/k_0)^2]}\right\}$$

and taking the absolute value, we see that the wave packet spreads as it moves:

$$\sqrt{2}\Delta x(t) \equiv L(t) = \sqrt{L^2+\frac{(v_gt/k_0)^2}{L^2}} = \sqrt{L^2+\frac{a^4\omega_0^2t^2}{L^2}}\ .$$

This is a general feature of non-linear Gaussian wave packets: for the same reason $(\omega_k = \sqrt{(m^2c^4/\hbar^2)+k^2})$ wave packets corresponding to relativistic particles broaden with time.

7.9 Causality between E and D and Kramers-Kronig Relations

When $\epsilon(\omega)$ is frequency dependent, there is a non-local temporal relation between **D** and **E**. To exhibit this, we write **D** and **E** in terms of their temporal Fourier components

$$\mathbf{D}(\mathbf{x},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathbf{D}}(\mathbf{x},\omega)e^{-i\omega t}. \tag{7.9.1}$$

For a linear medium

$$\tilde{\mathbf{D}}(\mathbf{x},\omega) = \epsilon(\omega)\tilde{\mathbf{E}}(\mathbf{x},\omega), \tag{7.9.2}$$

and thus

$$\mathbf{D}(\mathbf{x},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\epsilon(\omega)\tilde{\mathbf{E}}(\mathbf{x},\omega)e^{-i\omega t}. \tag{7.9.3}$$

We now use

$$\tilde{\mathbf{E}}(\mathbf{x},\omega) = \int_{-\infty}^{\infty}dt'\,\mathbf{E}(\mathbf{x},t')e^{+i\omega t'} \tag{7.9.4}$$

to get

$$\mathbf{D}(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\epsilon(\omega)e^{-i\omega t}\int_{-\infty}^{\infty}dt'\,e^{i\omega t'}\mathbf{E}(\mathbf{x},t'). \tag{7.9.5}$$

To display the non-locality, we write

$$\epsilon(\omega) = \epsilon_0 \left[\left(\frac{\epsilon(\omega)}{\epsilon_0} - 1 \right) + 1 \right] = \epsilon_0 [\chi_e(\omega) + 1] \tag{7.9.6}$$

and thus

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \left\{ \mathbf{E}(\mathbf{x}, t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, dt'\, e^{i\omega(t'-t)} \chi_e(\omega) \mathbf{E}(\mathbf{x}, t') \right\} . \tag{7.9.7}$$

Next step is to write $\chi_e(\omega)$ in Fourier representation

$$\chi_e(\omega) = \int_{-\infty}^{\infty} d\tau\, G(\tau) e^{i\omega\tau} . \tag{7.9.8}$$

The inverse transformation is

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \chi_e(\omega) e^{-i\omega\tau} . \tag{7.9.9}$$

As a result, we have

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \mathbf{E}(\mathbf{x}, t) + \epsilon_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, G(\tau) \underbrace{\int_{-\infty}^{\infty} dt'\, \mathbf{E}(\mathbf{x}, t') \underbrace{\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t'-t)} e^{i\omega\tau}}_{2\pi\delta(t'-t+\tau)}}_{2\pi\mathbf{E}(\mathbf{x}, t-\tau)} . \tag{7.9.10}$$

Integration over ω gives $2\pi\delta(t'-t+\tau)$. After the next integration over t' (which sets $t' = t-\tau$ in $\mathbf{E}(\mathbf{x}, t')$) we get

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \left\{ \mathbf{E}(\mathbf{x}, t) + \int_{-\infty}^{\infty} d\tau\, G(\tau) \mathbf{E}(\mathbf{x}, t-\tau) \right\} . \tag{7.9.11}$$

We have essentially re-derived here the convolution theorem of Fourier transforms, and have exhibited the non-local connection between **D** and **E**.

To explore the nature of this connection, we consider a simple one-resonance model for $\chi_e(\omega)$

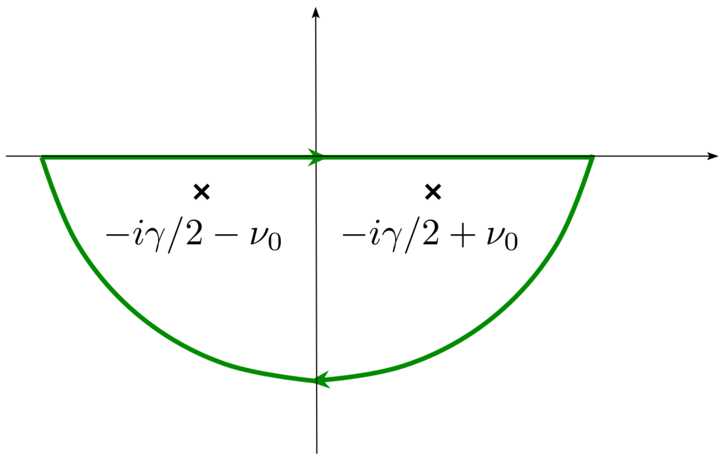
$$\chi_e(\omega) = \frac{\omega_P^2}{\omega_0^2 - \omega^2 - i\gamma\omega} = -\frac{\omega_P^2}{(\omega + i\gamma/2 + \nu_0)(\omega + i\gamma/2 - \nu_0)} , \tag{7.9.12}$$

where ω_P is the plasma frequency and

$$\nu_0^2 = \omega_0^2 - \gamma^2/4 . \tag{7.9.13}$$

This function has poles in the lower half plane at

$$\omega = -i\,\frac{\gamma}{2} \pm \nu_0 . \tag{7.9.14}$$



To evaluate $G(\tau)$ we use contour integration, noting that there are two cases

1. $\tau > 0$: integral over semicircle at $|\tau| = \infty$ vanishes in lower half plane.

2. $\tau < 0$: integral over semicircle at $|\tau| = \infty$ vanishes in upper half plane.

Thus $G(\tau)$ vanishes for $\tau < 0$. By the residue theorem

$$\begin{aligned} G(\tau > 0) &= -\frac{\omega_P^2}{2\pi} \times (-2\pi i) \times \sum_{\text{residues}} \dots \\ &= i\omega_P^2 \left[\frac{e^{-i\tau(-i\gamma/2+\nu_0)}}{2\nu_0} + \frac{e^{-i\tau(-i\gamma/2-\nu_0)}}{-2\nu_0} \right] = \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0}, \end{aligned} \tag{7.9.15}$$

and thus

$$G(\tau) = \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0} \theta(\tau). \tag{7.9.16}$$

We can make two observations

- There is an oscillatory frequency $\nu_0 \approx \omega_0$.

- The damping factor $1/\gamma$ is that of the oscillators.

Thus non-locality is confined to a region $\tau \approx \gamma^{-1}$.

7.9.1 Causality

Because $G(\tau)$ vanishes for $\tau < 0$, $\mathbf{D}$ only depends on the values of $\mathbf{E}$ at earlier times, i.e.

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \left[\mathbf{E}(\mathbf{x}, t) + \int_0^\infty d\tau\, G(\tau) \mathbf{E}(\mathbf{x}, t - \tau) \right]. \tag{7.9.17}$$

We can thus write the dielectric constant as

$$\epsilon(\omega)/\epsilon_0 = 1 + \int_0^\infty d\tau G(\tau) e^{i\omega\tau}. \tag{7.9.18}$$

Since $G(\tau)$ is real, we have

$$\epsilon(-\omega) = \epsilon^*(\omega^*). \tag{7.9.19}$$

Furthermore, if $G(\tau)$ is finite $\forall \tau$, $\epsilon(\omega)/\epsilon_0$ is analytic in the upper half plane, since integral is convergent there. We can therefore apply Cauchy's theorem for any z in the upper half plane

$$\epsilon(z)/\epsilon_0 - 1 = \frac{1}{2\pi i} \oint d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z}. \tag{7.9.20}$$

If we assume that $[\epsilon(\omega)/\epsilon_0 - 1]$ vanishes as $|\omega| \to \infty$ (in fact, in *Jackson* it is argued that $[\epsilon(\omega)/\epsilon_0 - 1] \sim 1/\omega^2$ for large ω), the contribution from the semi-circle at infinity vanishes, and we have

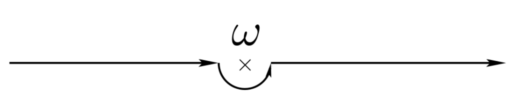
$$\epsilon(z)/\epsilon_0 - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z} \, . \tag{7.9.21}$$

We now consider a point just *above* the ω'-axis, by writing $z = \omega + i\delta$. Then

$$\frac{1}{\omega' - \omega - i\delta} = \mathrm{P}\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega) \tag{7.9.22}$$

whence

$$\epsilon(\omega)/\epsilon_0 - 1 = \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega}. \tag{7.9.23}$$



Thus taking the real and imaginary parts, we find

$$\begin{aligned} \mathrm{Re}\,\epsilon(\omega)/\epsilon_0 &= 1 + \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega'\,\frac{\mathrm{Im}\,\epsilon(\omega')/\epsilon_0}{\omega'-\omega} \\ \mathrm{Im}\,\epsilon(\omega)/\epsilon_0 &= -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega'\,\frac{\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1}{\omega'-\omega} \end{aligned} \tag{7.9.24}$$

These are the **Kronig-Kramers** relations; they relate **absorption** (*imaginary* part of ϵ) to **dispersion** (*real* part of ϵ) through analyticity.
In fact, on the real axis, the real part of $\epsilon(\omega)$ is an even function of (ω), while its imaginary part is an odd function of (ω). Indeed, for real ω, the relation $\epsilon(-\omega) = \epsilon^*(\omega^*)$ converts into

$$\mathrm{Re}\,\epsilon(-\omega) + i\,\mathrm{Im}\,\epsilon(-\omega) = \mathrm{Re}\,\epsilon(\omega) - i\,\mathrm{Im}\,\epsilon(\omega)\ , \tag{7.9.25}$$

which gives $\mathrm{Re}\,\epsilon(-\omega) = \mathrm{Re}\,\epsilon(\omega)$ and $\mathrm{Im}\,\epsilon(-\omega) = -\mathrm{Im}\,\epsilon(-\omega)$. This observation allows one to write the KK relations in terms of integrals over positive ω' only

$$\begin{aligned} \mathrm{Re}\,\epsilon/\epsilon_0 &= 1 + \frac{2}{\pi}\mathrm{P}\int_0^{\infty} d\omega'\,\frac{\omega'\,\mathrm{Im}\,\epsilon(\omega')/\epsilon_0}{\omega'^2-\omega^2} \\ \mathrm{Im}\,\epsilon/\epsilon_0 &= -\frac{2\omega}{\pi}\mathrm{P}\int_0^{\infty} d\omega'\,\frac{\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1}{\omega'^2-\omega^2}\ . \end{aligned} \tag{7.9.26}$$