Contents

Chapter 8

Wave Guides and Cavities

In this chapter we will consider propagation of waves in hollow, metal wave guides and cavities.

- *wave guide*: ends are *open*
- *cavity*: ends are *closed*

8.1 Boundary Conditions at Surface of Conductor

Recall that at the boundary between two media, 1 and 2, we have

$$\begin{aligned}
(\mathbf{H_2} - \mathbf{H_1}) \times \mathbf{n} &= \mathbf{K} \\
(\mathbf{B_2} - \mathbf{B_1}) \cdot \mathbf{n} &= 0 \\
(\mathbf{D_2} - \mathbf{D_1}) \cdot \mathbf{n} &= \sigma \\
(\mathbf{E_2} - \mathbf{E_1}) \times \mathbf{n} &= 0.
\end{aligned}$$

Inside a conductor, the electrons are completely free, with infinitely fast response, such that $\mathbf{B} = \mathbf{E} = 0$.
Thus our boundary conditions just below the conducting surface reduce to

$$\begin{aligned}
\mathbf{H} \times \mathbf{n} &= K \\
\mathbf{B} \cdot \mathbf{n} &= 0 \\
\mathbf{D} \cdot \mathbf{n} &= \sigma \\
\mathbf{E} \times \mathbf{n} &= 0.
\end{aligned}$$

Thus just outside the surface of the conductor, we have that

- **B** is **tangential** to the surface.
- **E** is **normal** to the surface.

The case where we do not have a **perfect** conductor is discussed in detail in *Jackson*, chapter 8.1. Note that in these cases we have energy losses associated with the absorption at the boundary surface.

8.2 Propagation of Monochromatic Wave

We consider the propagation of monochromatic waves in a hollow cylinder, of arbitrary cross section, which we take to be uniform along, say, the z-direction. We assume a harmonic time dependence $e^{-i\omega t}$, so that Maxwell's equations become

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega \mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= -i\mu\epsilon\omega \mathbf{E} \\ \nabla \cdot \mathbf{E} &= 0 \end{aligned}$$

Thus, in the usual way, these equations reduce to

$$(\nabla^2 + \mu\epsilon\omega^2) \left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right\} = 0 \tag{8.2.1}$$

Because of the cylindrical symmetry in the problem, we expect to find waves travelling in the positive or negative direction, or standing waves. Therefore we look for solutions of the form

$$\left. \begin{array}{c} \mathbf{E}(\mathbf{x}, t) \\ \mathbf{B}(\mathbf{x}, t) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{E}(x, y) \\ \mathbf{B}(x, y) \end{array} \right\} e^{\pm ikz - i\omega t}. \tag{8.2.2}$$

Note: this *does not mean* that the propagation vector is in the z direction as such.
We now write

$$\nabla^2 = \nabla_T^2 + \nabla_z^2 \tag{8.2.3}$$

where

$$\begin{aligned} \nabla_T^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ \nabla_z^2 &= \frac{\partial^2}{\partial z^2}. \end{aligned}$$

Then our wave equation (8.2.1) reduces to

$$[\nabla_T^2 + (\mu\epsilon\omega^2 - k^2)]\mathbf{E} = 0 \qquad (8.2.4)$$

and similarly for $\mathbf{B}$.

We now write $\mathbf{E}$ and $\mathbf{B}$ in terms of components parallel and transverse to z, i.e. $\mathbf{E} = \mathbf{E}_z + \mathbf{E}_T$ etc., and show that it is only necessary to solve for the longitudinal components E_z and B_z. We start with two of Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mathbf{B} \\ \nabla \times \mathbf{B} &= -i\mu\epsilon\omega\mathbf{E}. \end{aligned} \qquad (8.2.5)$$

Writing the first of these in terms of longitudinal and transverse components, we have

$$(\nabla_T + \nabla_z) \times (\mathbf{E}_T + \mathbf{E}_z) = i\omega(\mathbf{B}_T + \mathbf{B}_z). \qquad (8.2.6)$$

If we now consider the transverse and longitudinal components, we find

$$\nabla_T \times \mathbf{E}_T = i\omega\mathbf{B}_z \qquad (8.2.7)$$

$$\nabla_T \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_T = i\omega\mathbf{B}_T. \qquad (8.2.8)$$

Taking the ∇_z curl of the second of these equations, we find

$$\begin{aligned} i\omega\nabla_z \times \mathbf{B}_T &= \nabla_z \times [\nabla_T \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_T] \\ &= \nabla_T[\nabla_z \cdot \mathbf{E}_z] - \nabla_z^2\mathbf{E}_T \ . \end{aligned}$$

Then, using the z-dependence of $\mathbf{E}, \mathbf{B}$, we find

$$i\omega\nabla_z \times \mathbf{B}_T = \nabla_T[\nabla_z \cdot \mathbf{E}_z] + k^2\mathbf{E}_T. \qquad (8.2.9)$$

To proceed further, we use the second equation of (8.2.5), which becomes

$$\nabla_T \times \mathbf{B}_T = -i\mu\epsilon\omega\mathbf{E}_z \qquad (8.2.10)$$

$$\nabla_T \times \mathbf{B}_z + \nabla_z \times \mathbf{B}_T = -i\mu\epsilon\omega\mathbf{E}_T \ . \qquad (8.2.11)$$

Substituting $\nabla_z \times \mathbf{B}_T = -i\mu\epsilon\omega\mathbf{E}_T - \nabla_T \times \mathbf{B}_z$ in Eq. (8.2.9), we find

$$i\omega[-i\mu\epsilon\omega\mathbf{E}_T - \nabla_T \times \mathbf{B}_z] = k^2\mathbf{E}_T + \nabla_T[\nabla_z \cdot \mathbf{E}_z] \ . \qquad (8.2.12)$$

Note that $\mathbf{B}_z = \mathbf{e_z}B_z$, and hence $\nabla_T \times \mathbf{B}_z = -\mathbf{e_z} \times \nabla_T B_z$. Then we have

$$i\omega[-i\mu\epsilon\omega\mathbf{E}_T + \mathbf{e_z} \times \nabla_T B_z] = k^2\mathbf{E}_T + \nabla_T[\nabla_z \cdot \mathbf{E}_z] \ . \qquad (8.2.13)$$

or

$$(\mu\epsilon\omega^2 - k^2)\mathbf{E}_T = -i\omega\mathbf{e_z} \times \nabla_T B_z + \nabla_T[\nabla_z \cdot \mathbf{E}_z] \qquad (8.2.14)$$

and

$$\mathbf{E}_T = (\mu\epsilon\omega^2 - k^2)^{-1}[\nabla_T(\nabla_z \cdot \mathbf{E}_z) - i\omega\mathbf{e_z} \times \nabla_T B_z] \ . \qquad (8.2.15)$$

Thus we can see that we have expressed the **transverse** components $\mathbf{E}_T$ entirely in terms of **longitudinal** components E_z and B_z.
Similarly, writing Eqs.(8.2.10),(8.2.11) as

$$\nabla_T \times \mathbf{H}_T = -i\epsilon\omega\mathbf{E}_z \ , \qquad (8.2.16)$$

$$\nabla_T \times \mathbf{H}_z + \nabla_z \times \mathbf{H}_T = -i\epsilon\omega\mathbf{E}_T \ , \qquad (8.2.17)$$

and comparing these equations with Eqs.(8.2.7), (8.2.8), we obtain

$$\mathbf{H}_T = (\mu\epsilon\omega^2 - k^2)^{-1}[\nabla_T(\nabla_z \cdot \mathbf{H}_z) + i\epsilon\omega\mathbf{e_z} \times \nabla_T E_z] \ . \qquad (8.2.18)$$

Again, **transverse** components $\mathbf{H}_T$ are expressed entirely in terms of **longitudinal** components E_z and H_z.

8.3 Classification of Modes

We have now shown that the propagation of the waves can be described solely by solving the two-dimensional wave equation

$$(\nabla_T^2 + \mu\epsilon\omega^2 - k^2)\left\{ \begin{array}{c} E_z(x,y) \\ B_z(x,y) \end{array} \right\} = 0, \qquad (8.3.1)$$

subject to suitable boundary conditions. In the case of perfectly conducting walls S, the boundary conditions are

$$\mathbf{n} \times \mathbf{E}|_S = 0$$

$$\mathbf{n} \cdot \mathbf{B}|_S = 0 \ .$$

It can be shown that these boundary conditions are equivalent to

$$E_z|_S = 0 \qquad (8.3.2)$$

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0. \qquad (8.3.3)$$

The first of these conditions is trivial.
To get the second we recall the Maxwell's equation

$$\nabla \times \mathbf{B} \;=\; -i\mu\epsilon\omega\mathbf{E}\,, \tag{8.3.4}$$

whose transverse part is (see Eq. (8.2.11))

$$\nabla_T \times \mathbf{B}_z + \nabla_z \times \mathbf{B}_T \;=\; -i\mu\epsilon\omega\mathbf{E}_T\ , \tag{8.3.5}$$

or using $\mathbf{B}_z = \mathbf{e_z} B_z$,

$$-\mathbf{e}_z \times \nabla_T B_z + \mathbf{e}_z \times \frac{\partial}{\partial z}\mathbf{B}_T = -i\mu\epsilon\omega\mathbf{E}_T\ .$$

Forming its cross-product with $\mathbf{e}_z$, we have

$$\nabla_T B_z - \frac{\partial}{\partial z}\mathbf{B}_T \;=\; -i\mu\epsilon\omega\,\mathbf{e}_z \times \mathbf{E}_T\ . \tag{8.3.6}$$

Then, taking the component of this equation along $\mathbf{n}$ we get

$$(\mathbf{n}\cdot\nabla_T)B_z - \frac{\partial}{\partial z}(\mathbf{n}\cdot\mathbf{B}_T) \;=\; -i\mu\epsilon\omega(\mathbf{e}_z \times \mathbf{E}_T)\cdot\mathbf{n} \tag{8.3.7}$$

or, using the cycling property $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ of the mixed product

$$\frac{\partial B_z}{\partial n} - \frac{\partial B_n}{\partial z} = -i\mu\epsilon\omega\,\mathbf{e}_z\cdot(\mathbf{E}_T \times \mathbf{n})\ . \tag{8.3.8}$$

On the conducting wall S we have $B_n|_S = 0$ for all z, hence $\partial B_n/\partial z|_S = 0$. Also $\mathbf{E}\times\mathbf{n}|_S = 0$, and we obtain $\partial B_z/\partial n|_S = 0$, i.e. the boundary condition (8.3.3).
In principle, we are simultaneously solving two boundary-value equations subject to each of the above conditions. However, in general the eigenvalue equation (8.2.4) will have *different* eigenvalues for the two different sets of boundary conditions. Hence we cannot satisfy both simultaneously **unless one is trivial**. Thus we classify the solutions as

Transverse Magnetic (TM)

Here $B_z = 0$ everywhere, and $E_z = 0$ on boundary. The differential equation (8.3.1)a with the above Dirichlet boundary condition determines E_z in the wave guide. If we know E_z, the transverse fields can be obtained from Eq. (8.2.15),

$$\mathbf{E}_T \;=\; (\mu\epsilon\omega^2 - k^2)^{-1}[\nabla_T(\nabla_z\cdot\mathbf{E}_z) - i\omega\mathbf{e_z}\times\nabla_T B_z]$$

that gives

$$\mathbf{E}_T \;=\; \frac{ik}{\gamma^2}\vec{\nabla}_T E_z, \tag{8.3.9}$$

and Eq. (8.2.18)

$$\mathbf{H}_T = (\mu\epsilon\omega^2 - k^2)^{-1}[\nabla_T(\nabla_z \cdot \mathbf{H}_z) + i\epsilon\omega\mathbf{e_z} \times \nabla_T E_z] \ ,$$

that gives

$$\mathbf{H}_T = \frac{i\epsilon\omega}{\gamma^2}\mathbf{e}_z \times \vec{\nabla}_T E_z \ . \tag{8.3.10}$$

We introduced here the notation $\gamma^2 = \mu\epsilon\omega^2 - k^2$.

Transverse Electric (TE)

$E_z = 0$ everywhere, and $\partial B_z/\partial n = 0$ on boundary. Here we must solve Eq. (8.3.1)b with Neumann boundary condition. The transverse fields are

$$\mathbf{E}_T = -\frac{i\mu\omega}{\gamma^2}\mathbf{e}_z \times \vec{\nabla}_T H_z, \quad \mathbf{H}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T H_z \ . \tag{8.3.11}$$

Finally, we must consider

Transverse Electric Magnetic (TEM)

Here we have $B_z = E_z = 0$ everywhere, so that the only non-trivial components are those in the transverse direction. Then Maxwell's equations reduce to

$$\begin{aligned} \nabla_T \times \mathbf{E}_{\text{TEM}} &= 0 \\ \nabla_z \times \mathbf{E}_{\text{TEM}} &= i\omega\mathbf{B}_{\text{TEM}}. \end{aligned}$$

In addition, we have

$$\nabla_T \cdot \mathbf{E}_{\text{TEM}} = 0 \ . \tag{8.3.12}$$

Combining the first and third of these equations, we find

$$\nabla_T \times (\nabla_T \times \mathbf{E}_{\text{TEM}}) = 0 \quad \Rightarrow \quad \nabla_T \underbrace{(\nabla_T \cdot \mathbf{E}_{\text{TEM}})}_{0} - \nabla_T^2 \mathbf{E}_{\text{TEM}} = 0 \quad \Rightarrow \quad \nabla_T^2 \mathbf{E}_{\text{TEM}} = 0, \tag{8.3.13}$$

and comparing with the wave equation (8.2.4), we find

$$k^2 = \mu\epsilon\omega^2. \tag{8.3.14}$$

This is just the **infinite-medium value**. Similarly, we find

$$i\omega\mathbf{B}_{\text{TEM}} = ik\mathbf{e}_z \times \mathbf{E}_{\text{TEM}} \quad \Rightarrow \quad \mathbf{B}_{\text{TEM}} = \pm\sqrt{\mu\epsilon}\, \mathbf{e_z} \times \mathbf{E}_{\text{TEM}}. \tag{8.3.15}$$

Thus we essentially have **plane-wave propagation**.
We see that $\mathbf{E}_{\text{TEM}}$ obeys Laplace's equation. Furthermore, the walls of the wave guide are an equipotential. Thus the only solution inside a single, hollow perfect conductor is the trivial one.

TEM modes cannot propagate inside a single conductor

They can, however, propagate inside a coaxial cable.

8.4 Modes of a Waveguide

We begin by discussing TM modes, for which we write

$$E_z = \phi(x, y)e^{\pm ikz - i\omega t}. \tag{8.4.1}$$

Then ϕ satisfies

$$(\nabla_T^2 + \mu\epsilon\omega^2 - k^2)\phi = 0, \tag{8.4.2}$$

subject to $\phi = 0$ on the boundary.
We now introduce

$$\gamma^2 = \mu\epsilon\omega^2 - k^2, \tag{8.4.3}$$

so that our eigenvalue equation becomes

$$(\nabla_T^2 + \gamma^2)\phi = 0. \tag{8.4.4}$$

In general, the boundary conditions require that γ^2 be positive, yielding a discrete set of eigenvalues $\{\gamma_\lambda\}$, with corresponding wave number k_λ satisfying $\mu\epsilon\omega^2 - k_\lambda^2 = \gamma_\lambda^2$ or

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2. \tag{8.4.5}$$

If $k_\lambda^2 > 0$, k_λ is real, and the propagation is oscillatory. If it is negative, the wave number is imaginary and the wave will not propagate.
We define the cut-off frequency ω_λ by

$$\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}} \ . \tag{8.4.6}$$

Then we deal with two cases:

- $\omega < \omega_\lambda$: wave cannot propagate
- $\omega > \omega_\lambda$: wave can propagate

Finally, it is worth noting that the group velocity of the wave in the wave guide is always smaller than the speed of light. We first note that we may write

$$k_\lambda = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2}. \tag{8.4.7}$$

We recall that the **phase velocity**

$$\begin{aligned} v_p &= \omega/k \\ &= \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{\sqrt{1-\omega_\lambda^2/\omega^2}} \\ &= \frac{c}{\sqrt{1-\omega_\lambda^2/\omega^2}} \end{aligned}$$

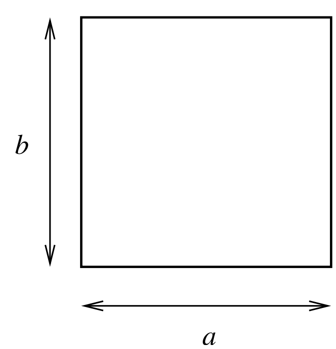
which is always **larger** than the velocity of light, and diverges as $\omega \to \omega_\lambda$.
In contrast, the **group velocity**

$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = c\sqrt{1-\omega_\lambda^2/\omega^2}, \tag{8.4.8}$$

which is always **smaller** than the infinite-space velocity of light, and vanishes as $\omega \to \omega_\lambda$. In this limit the *wave no longer propagates*. Note that

$$v_p v_g = c^2. \tag{8.4.9}$$

8.5 Modes of a Rectangular Waveguide



For the sake of illustration, we will consider the case of **TE modes**. In Cartesian coordinates, we have to solve the eigenvalue equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right]\psi = 0 \tag{8.5.1}$$

subject to

$$\begin{aligned}\frac{\partial\psi(0,y)}{\partial x} = \frac{\partial\psi(a,y)}{\partial x} &= 0,\\ \frac{\partial\psi(x,0)}{\partial y} = \frac{\partial\psi(x,b)}{\partial y} &= 0.\end{aligned}$$

This clearly has eigenfunctions for H_z

$$\psi_{mn}(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{8.5.2}$$

with eigenvalues

$$\gamma_{mn}^2 = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]. \tag{8.5.3}$$

We denote the modes $\mathrm{TE}_{m,n}$. The lowest non-trivial mode is $\mathrm{TE}_{1,0}$ if $a > b$, with cut-off frequency given by

$$\gamma_{10}^2 = \pi^2/a^2. \tag{8.5.4}$$

For this mode, for wave propagating in the positive direction, we have

$$H_z = H_0\cos\left(\frac{\pi x}{a}\right)e^{izk_{1,0}-i\omega t}. \tag{8.5.5}$$

We can obtain the transverse components of the field from Eq. (8.3.11)

$$\mathbf{E}_T \;=\; -\frac{i\mu\omega}{\gamma^2}\mathbf{e}_z\times\vec{\nabla}_T H_z, \quad \mathbf{H}_T \;=\; \frac{ik}{\gamma^2}\vec{\nabla}_T H_z\ .$$

that gives

$$\begin{aligned}\mathbf{H_T} &= -\frac{ika}{\pi}H_0\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}\mathbf{e_x}\\ \mathbf{E_T} &= \frac{i\omega a\mu}{\pi}H_0\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}\mathbf{e_y},\end{aligned}$$

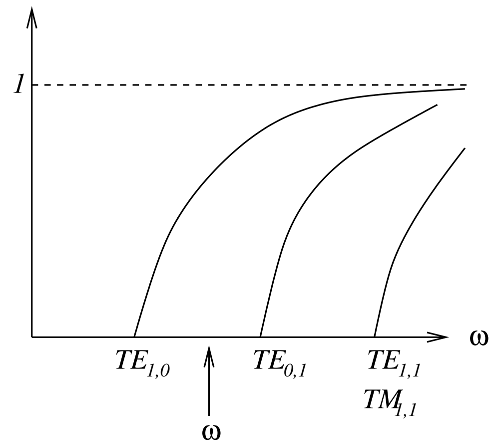
with $k = k_{1,0}$.

In this particular case, magnetic field has non-zero components in z-direction, $H_z \neq 0$ and in x-direction, $H_x \neq 0$, while $H_y = 0$. Using explicit expressions, it is easy to verify that $\vec{\nabla}\cdot\mathbf{H} = 0$, and in this sense $\mathbf{H}$ satisfies the transversality condition. For electric field, we have only one non-zero component, namely E_y, which is a function of x and z only, i.e. $\partial E_y/\partial y = 0$, and hence $\vec{\nabla}\cdot\mathbf{E} = 0$.

The analysis of TM modes proceeds likewise. However, here the lowest propagating mode is $\mathrm{TM}_{1,1}$, with a higher cut-off frequency. Wave guides are often constructed such that $\mathrm{TE}_{1,0}$ is the *only* propagating mode. Recalling that

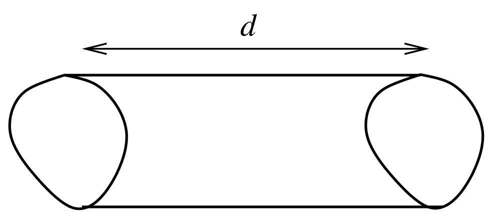
$$k_\lambda = \sqrt{\mu\epsilon\left(\omega^2 - \omega_\lambda^2\right)} \tag{8.5.6}$$

we can show $k_\lambda/\sqrt{\mu\epsilon\omega^2}$ as follows:



8.6 Resonant Cavities

A resonant cavity differs from a wave guide in being closed. Thus, rather than having wave propagation, we have standing waves.



As before, we can have both TM and TE fields. However, now the z-dependence is of the form, for the case of **TM modes**,

$$E_z = \phi(x,y)[A\sin kz + B\cos kz] \quad (8.6.1)$$

$$H_z = 0 \quad (8.6.2)$$

Then the transverse part of the wave is

$$\mathbf{E_T} = \frac{1}{\gamma^2}\nabla_T(\nabla_z \cdot \mathbf{E_z}) \quad (8.6.3)$$

$$= \frac{k}{\gamma^2}\nabla_T\phi(x,y)[A\cos kz - B\sin kz]. \quad (8.6.4)$$

Now the boundary condition $\mathbf{E_T} = 0$ at $z = 0, z = d$ yields $A = 0, k = p\pi/d$ and thus

$$E_z = \phi(x,y)\cos\frac{p\pi z}{d} \quad (8.6.5)$$

$$\mathbf{E_T} = -\frac{p\pi}{d\gamma^2}\sin\frac{p\pi z}{d}\nabla_T\phi. \quad (8.6.6)$$

We can obtain $\mathbf{H_T}$ similarly, yielding

$$\mathbf{H_T} \;=\; \frac{i\epsilon\omega}{\gamma^2}\mathbf{e}_z \times \vec{\nabla}_T E_z = \frac{i\epsilon\omega}{\gamma^2}\cos\frac{p\pi z}{d}\mathbf{e_z}\times\nabla_T\phi. \tag{8.6.7}$$

A corresponding analysis for the **TE modes** yields

$$H_z = \psi(x,y)(A\sin kz + B\cos kz)$$

so

$$\mathbf{E}_T \;=\; -\frac{i\mu\omega}{\gamma^2}\mathbf{e}_z\times\vec{\nabla}_T H_z = -\frac{i\omega\mu}{\gamma^2}(A\sin kz + B\cos kz)\,\mathbf{e}_z\times\nabla_T\psi\ .$$

From the boundary conditions $\mathbf{E}_T|_{z=0,d} = 0$ we get

$$\begin{aligned} H_z &= \psi(x,y)\sin\frac{p\pi z}{d} \\ \mathbf{E_T} &= -\frac{i\omega\mu}{\gamma^2}\sin\frac{p\pi z}{d}\,\mathbf{e_z}\times\nabla_T\psi \\ \mathbf{H_T} &= \frac{1}{\gamma^2}\nabla_T(\nabla_z\cdot\mathbf{H}_z) = \frac{p\pi}{d\gamma^2}\cos\frac{p\pi z}{d}\nabla_T\psi. \end{aligned} \tag{8.6.8}$$

The function $\psi(x,y)$ now satisfies the wave equation

$$\nabla_T^2\psi + \gamma^2\psi = 0 \tag{8.6.9}$$

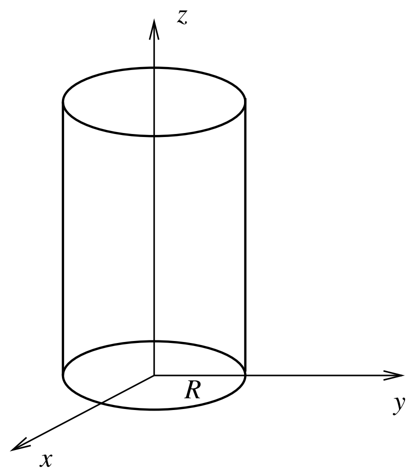
where

$$\gamma^2 = \mu\epsilon\omega^2 - \frac{p^2\pi^2}{d^2}. \tag{8.6.10}$$

We can solve this eigenvalue problem as for propagation along a wave guide, but now the eigenvalues γ_λ determine not the cut-off frequencies but the **allowed frequencies**:

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left[\gamma_\lambda^2 + \frac{p^2\pi^2}{d^2}\right] \tag{8.6.11}$$

Example: cylindrical cavity, radius R



We work in cylindrical polar coords $\psi(s,\varphi)$. Because of cylindrical symmetry, we seek separable solutions to the two-dimensional wave equation of the form

$$\psi(s,\varphi) = \psi(s)e^{\pm im\varphi} \tag{8.6.12}$$

where $m = 0, 1, 2, \ldots$. Then we have

$$\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \gamma^2 - \frac{m^2}{s^2}\right)\psi(s) = 0. \tag{8.6.13}$$

This is just Bessel's equation (see last semester), with solution

$$\psi(s,\varphi) = J_m(\gamma_{mn}s)e^{\pm im\varphi}. \tag{8.6.14}$$

In the case of a **TM mode**, where $\psi(s,\varphi) = 0$ at $s = R$, we have

$$\gamma_{mn}R = x_{mn}, \tag{8.6.15}$$

where x_{mn} is the n^{th} root of $J_m(x) = 0$. Thus the resonant frequencies are given by

$$\omega^2_{mnp} = \frac{1}{\mu\epsilon}\left[\frac{x^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}\right] \qquad \text{(TM mode)}. \tag{8.6.16}$$

The solution for **TE modes** is similar and the resonant frequencies are given by

$$\omega^2_{mnp} = \frac{1}{\mu\epsilon}\left[\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}\right] \qquad \text{(TE mode)}, \tag{8.6.17}$$

where x'_{mn} is now the n^{th} root of $J'_m(x) = 0$.
Note that for TM modes we have $p = 0, 1, 2, \ldots$ whilst for TE modes we have $p = 1, 2, 3, \ldots$. However, the smallest x'_{mn} is $x'_{11} = 1.841...$ while the smallest x_{mn} is $x_{01} = 2.405...$, and thus for sufficiently large d the dominant mode is

$$\text{TE}_{1,1,1}. \tag{8.6.18}$$

8.7 Energy Flux along Waveguide

The time-averaged energy flux is given by the *real part* of the **Poynting Vector**

$$\mathbf{S} = \frac{1}{2}\,\mathbf{E} \times \mathbf{H}^*. \tag{8.7.1}$$

Let us evaluate this for **TE** modes

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = -\frac{1}{2}\Big(\mathbf{H}_T^* \times \mathbf{E}_T + H_z^* \mathbf{e}_z \times \mathbf{E}_T\Big). \tag{8.7.2}$$

Since $H_z = \psi(x, y)e^{-i\omega t + ikz}$, using

$$\mathbf{H}_T^* \;=\; -\frac{ik}{\gamma^2}\vec{\nabla}_T H_z^*, \quad \mathbf{E}_T \;=\; -\frac{i\mu\omega}{\gamma^2}\mathbf{e}_z \times \vec{\nabla}_T H_z \;. \tag{8.7.3}$$

(see Eq. (8.3.11)) we get

$$\mathbf{S} = \frac{\omega k\mu}{2\gamma^4}\nabla_T H_z^* \times (\mathbf{e}_z \times \nabla_T H_z) + \frac{i\omega\mu}{2\gamma^2}H_z^*\mathbf{e}_z \times (\mathbf{e}_z \times \nabla_T H_z) \;=\; \frac{\omega k\mu}{2\gamma^4}\mathbf{e}_z|\nabla_T\psi|^2 - i\frac{\omega\mu}{2\gamma^2}\psi^*\nabla_T\psi$$

Taking the real part, we get

$$\mathrm{Re}\,\mathbf{S} = \frac{\omega k\mu}{2\gamma^4}|\nabla_T\psi|^2\mathbf{e}_z. \tag{8.7.4}$$

This is in the z-direction, and we see that energy propagation is along the waveguide. Similarly, for the **TM** wave $E_z = \phi(x, y)e^{-i\omega t+ikz}$ one obtains

$$\mathrm{Re}\,\mathbf{S} = \frac{\omega k\epsilon}{2\gamma^4}|\nabla_T\phi|^2\mathbf{e}_z. \tag{8.7.5}$$

The total power transmitted by the **TE** wave is

$$P = \mathrm{Re}\int_A \mathbf{S}\cdot\mathbf{e_z}\,dA = \frac{\omega k\mu}{2\gamma^4}\int dA\,(\nabla_T\psi)^*\cdot(\nabla_T\psi). \tag{8.7.6}$$

where A is a cross-section through the wave guide. Recalling Green's identity, we have

$$\int(\psi^*\nabla_T^2\psi + \nabla_T\psi^*\cdot\nabla_T\psi)\,dA = \oint_C \psi^*\frac{\partial\psi}{\partial n}dl. \tag{8.7.7}$$

Because of the boundary conditions, either $\frac{\partial\psi}{\partial n}$ or ϕ (for the TM mode) vanish on the surface. Thus

$$\begin{aligned} P \;&=\; -\frac{\omega k\mu}{2\gamma^4}\int_A \psi^*\nabla_T^2\psi\,dA \\ &=\; \frac{\omega k\mu}{2\gamma^4}\gamma^2\int_A |\psi|^2\,dA \;, \end{aligned}$$

using wave equation

$$(\nabla_T^2 + \gamma^2)\psi = 0. \tag{8.7.8}$$

Thus we have

$$P = \frac{\mu}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \int_A \psi^* \psi \, dA, \tag{8.7.9}$$

where we represented k as $\omega\sqrt{\mu\epsilon}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}$ and γ^2 as $\mu\epsilon\omega_\lambda^2$.
Similarly, for the **TM** modes we get

$$P = \frac{\epsilon}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \int_A \phi^* \phi \, dA, \tag{8.7.10}$$

From Chapter 7, we have that the field energy per unit length is given by (for **TE** modes)

$$\begin{aligned} \langle U \rangle &= \frac{1}{4} \int [\epsilon \mathbf{E} \cdot \mathbf{E}^* + \mu \mathbf{H} \cdot \mathbf{H}^*] \, dA \;=\; \frac{1}{4} \int [\epsilon \mathbf{E}_T \cdot \mathbf{E}_T^* + \mu \mathbf{H}_T \cdot \mathbf{H}_T^* + \mu H_z \cdot H_z^*] \, dA \\ &= \frac{\mu}{4} \int \left[\frac{\mu\epsilon\omega^2 + k^2}{\gamma^4} |\nabla_T \psi|^2 + |\psi|^2 \right] dA \;=\; \frac{\mu}{4\gamma^2} (\mu\epsilon\omega^2 + k^2 + \gamma^2) \int |\psi|^2 \, dA \end{aligned}$$

where we have used the fact that $\int |\nabla_T \psi|^2 = \gamma^2 \int |\psi|^2$ since $\nabla_T^2 \psi = -\gamma^2 \psi$. Finally, we obtain

$$\langle U \rangle = \frac{\mu^2 \epsilon \omega^2}{2\gamma^2} \int |\psi|^2 dA = \frac{\mu}{2} \frac{\omega^2}{\omega_\lambda^2} \int |\psi|^2 \, dA. \tag{8.7.11}$$

Using Eqs. (8.7.9) and (8.7.11), we find

$$\boxed{P/U = \frac{1}{\sqrt{\mu\epsilon}} \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \equiv v_g} \tag{8.7.12}$$

Thus we see that the **energy propagates** with the **group velocity**.
N.B. you should convince yourself that this expression has the correct dimension.
For the **TM** wave, we get

$$\langle U \rangle = \frac{\epsilon}{2} \frac{\omega^2}{\omega_\lambda^2} \int |\phi|^2 \, dA.$$

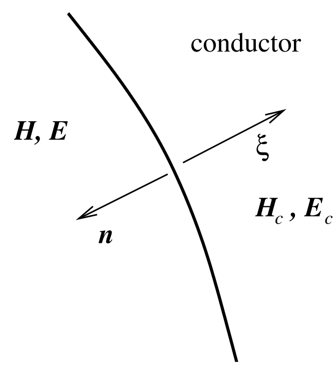
yielding the same result (8.7.12) for group velocity.

8.8 Boundary Conditions at Surface of Good Conductor

At surface of infinitely good conductor, we have

$$\begin{aligned} \mathbf{n}\cdot\mathbf{B} &= 0 \\ \mathbf{n}\times\mathbf{E} &= 0 \\ \mathbf{n}\cdot\mathbf{D} &= \Sigma \\ \mathbf{n}\times\mathbf{H} &= \mathbf{K} \end{aligned} \tag{8.8.1}$$

where Σ is the surface charge density, and $\mathbf{K}$ is the surface current density. These surface densities reflect the fact that there is no electric or magnetic field inside a conductor. The fields are nonzero outside a conductor, but quickly vanish inside a thin layer within a perfect conductor.



In the case of a conductor of finite conductivity σ, we have

$$\mathbf{J} = \sigma\mathbf{E}, \tag{8.8.2}$$

and the width of the layer is finite. To study the behavior of the fields inside the layer, we should take

$$\mathbf{n}\times(\mathbf{H}-\mathbf{H_c}) = 0, \tag{8.8.3}$$

where we use the subscript c to denote fields inside the conductor. (As $\sigma\to\infty$, we recover our surface current as a volume current over the thin layer close to the boundary).
We obtain the results for finite conductivity by successive approximation. We assume that initially $\mathbf{E}$ is perpendicular, and $\mathbf{H}$ parallel, to the surface just outside the conductor. Then $\mathbf{H}_c|_{\text{surface}} \simeq \mathbf{H}_\parallel$, and Maxwell's equations within the conductor become

$$\begin{aligned} \nabla\times\mathbf{E_c} + \mu_c\frac{\partial\mathbf{H}_c}{\partial t} &= 0 \\ \nabla\times\mathbf{H}_c = \mathbf{J} + \frac{\partial\mathbf{D}_c}{\partial t}\ . \end{aligned}$$

If we assume harmonic time dependence, these reduce to

$$\begin{aligned}\mathbf{H_c} &= -\frac{i}{\mu_c\omega}\nabla\times\mathbf{E_c}\\ \nabla\times\mathbf{H_c} &= \sigma\mathbf{E_c} - i\omega\epsilon\mathbf{E_c}.\end{aligned}$$

Thus if σ is sufficiently large, these reduce to

$$\begin{aligned}\mathbf{H_c} &= -\frac{i}{\mu_c\omega}\nabla\times\mathbf{E_c}\\ \mathbf{E_c} &= \frac{1}{\sigma}\nabla\times\mathbf{H_c}.\end{aligned}$$

We now assume all variation to be normal to the surface. (Spatial variation of the fields on the normal direction is much more rapid than in the parallel direction so we can neglect $\nabla_\parallel$ in comparison to ∇_T). Then we have

$$\nabla = -\mathbf{n}\frac{\partial}{\partial\xi} \tag{8.8.4}$$

and our equations become

$$\begin{aligned}\mathbf{H_c} &= \frac{i}{\mu_c\omega}\mathbf{n}\times\frac{\partial\mathbf{E_c}}{\partial\xi}\\ \mathbf{E_c} &= -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial\mathbf{H_c}}{\partial\xi}.\end{aligned}$$

We immediately see that $\mathbf{n}\cdot\mathbf{H_c} = 0$, consistent with our boundary assumptions. Furthermore, combining these two equations we obtain

$$\mathbf{H_c} = -\frac{i}{\mu_c\omega\sigma}\mathbf{n}\times\left[\mathbf{n}\times\frac{\partial^2\mathbf{H_c}}{\partial\xi^2}\right], \tag{8.8.5}$$

yielding

$$\frac{\partial^2}{\partial\xi^2}\mathbf{H_c} + \frac{2i}{\delta^2}\mathbf{H_c} = 0, \tag{8.8.6}$$

where

$$\delta \equiv \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2}, \tag{8.8.7}$$

is the skin depth. Thus, combining this with the condition $\mathbf{n}\cdot\mathbf{H_c} = 0$, we find

$$\mathbf{H_c} = \mathbf{H}_\parallel e^{(i-1)\xi/\delta}. \tag{8.8.8}$$

Thus the magnetic field is **tangential** and falls off **exponentially** as we go into the conductor. We can differentiate this, to obtain

$$\mathbf{E_c} = \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\mathbf{n}\times\mathbf{H}_{\parallel})e^{-\xi/\delta}e^{i\xi/\delta}. \tag{8.8.9}$$

Thus $\mathbf{E_c}$ is also tangential to the surface, but of much smaller magnitude.
We now go back to our boundary condition

$$\mathbf{n}\times(\mathbf{E}-\mathbf{E_c}) = 0. \tag{8.8.10}$$

Since $\mathbf{E}_c$ has a small tangential component, so does $\mathbf{E}$ just outside the conductor.

$$\mathbf{E}_{\parallel} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)(\mathbf{n}\times\mathbf{H}_{\parallel}) = \frac{1}{\sigma\delta}(1-i)(\mathbf{n}\times\mathbf{H}_{\parallel}).$$

Thus there is a non-zero component of the Poynting vector into the conductor, and hence a net flow of energy, given by

$$\begin{aligned}\left\langle\frac{dP}{da}\right\rangle &= \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot(-\mathbf{n}) \\ &= \frac{1}{2\sigma\delta}|\mathbf{H}_{\parallel}|^2 \\ &= \frac{\mu_c\omega\delta}{4}|\mathbf{H}_{\parallel}|^2 \ .\end{aligned}$$

It can be demonstrated that this power is dissipated into heat as ohmic losses in the skin of the conductor.
Applying this to our wave guide, we see that we have an energy loss/unit length given by

$$\begin{aligned}\frac{dP}{dz} &= -\frac{1}{2\sigma\delta}\oint_C dl\,|\mathbf{H}_{\parallel}|^2 = -\frac{1}{2\sigma\delta}\oint_C dl\,|\mathbf{n}\times\mathbf{H}|^2 \\ &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2\oint_C dl\left\{\begin{array}{lr}\frac{1}{\mu^2\omega_\lambda^2}\left|\frac{\partial\phi}{\partial n}\right|^2 & \text{(TM)} \\ \frac{1}{\mu\epsilon\omega_\lambda^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)|\mathbf{n}\times\nabla_T\psi|^2 + \frac{\omega_\lambda^2}{\omega^2}|\psi|^2 & \text{(TE)}\end{array}\right.\end{aligned}$$