804 Final Exam (40 points). 05/02/12, 12:00 - 15:00

Problem 1

A circularly polarized electromagnetic plane wave is normally incident on an infinitely large plane made from a perfect conductor. Find the charge and current densities induced on the conducting plane.

Solution

Let us choose the z axis in the direction normal to the plane and the wave coming from above the plane. The incident wave has the form ($\hat{e}_3 \times \hat{e}_+ = -i\hat{e}_+$)

$$\vec{E} \;=\; -i\hat{e}_+ E_0 e^{-i\omega t + ikz}, \quad \vec{B} \;=\; \frac{\hat{n}}{c} \times \vec{E} \;=\; -\frac{\hat{e}_3}{c} \times \vec{E} \;=\; \hat{e}_+ \frac{E_0}{c} e^{-i\omega t + ikz}$$

where $\hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)$ and $-i$ is added for convenience (E_0 is real). The reflected wave is

$$\vec{E} \;=\; \vec{E}_R e^{-i\omega t - ikz}, \quad \vec{B} \;=\; \frac{\hat{n}}{c} \times \vec{E} \;=\; \frac{\hat{e}_3}{c} \times \vec{E}_R \; e^{-i\omega t - ikz}$$

The boundary condition for the electric field is $E_{\parallel}^{\rm above} = E_{\parallel}^{\rm below} = 0$ so $\vec{E}_R = i\hat{e}_+ E_0$ and the sum of the reflected and incident waves takes the form:

$$\vec{E} \;=\; 2\hat{e}_+ E_0 \sin kz \; e^{-i\omega t}, \quad \vec{B} \;=\; 2\hat{e}_+ \frac{E_0}{c} \cos kz \; e^{-i\omega t}$$

Now, σ and K can be found from the remaining boundary conditions

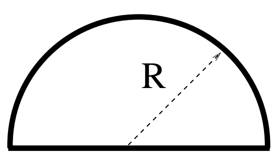
$$\epsilon_1 \vec{E}_{\perp}^{\rm above} - \epsilon_2 \vec{E}_{\perp}^{\rm below} = \sigma, \quad \frac{1}{\mu_1} \vec{B}_{\perp}^{\rm above} - \frac{1}{\mu_2} \vec{B}_{\perp}^{\rm below} = \vec{K}$$

We get $\sigma = 0$ and $\vec{K} \;=\; \frac{2}{\mu_0 c} \hat{e}_+ E_0 \; e^{-i\omega t}$. Taking the real part we get

$$K_x \;=\; \frac{2}{\mu_0 c} E_0 \; \cos \omega t, \quad K_y \;=\; \frac{2}{\mu_0 c} E_0 \; \sin \omega t,$$

Problem 2.

Find the cutoff frequency for the **TM** modes propagating in the cylindrical wave guide with the half-moon cross section (the roots of Bessel functions may be found in *Jackson*).



Hint: Look at the solutions for cylindrical wave guide and think about the symmetries of the system.

Solution

The solution for the TM mode in the case of cylindrical wave guide is

$$\psi(s, \theta) = J_m(x_{mn} \frac{s}{R}) e^{\pm im\phi}$$

For the half-moon cross section, we need an additional boundary condition $\psi(\theta = 0) = \psi(\theta = \pi) = 0$ which corresponds to

$$\psi(s, \theta) = J_m(x_{mn} \frac{s}{R}) \sin m\phi$$

Thus, the cutoff frequency corresponding to $m = 1$ is

$$\omega_{11} = c\frac{x_{11}}{R} = \frac{3.832}{R}c$$

Problem 3.

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the x, y plane, centered at the origin and at time $t = 0$, the charge is at $(a, 0)$ on the positive x axis. For points on the z axis, find
(a) the Lienard-Wiechert potentials and
(b) the time-averaged electric field.

Solution

(a). The Lienard-Wiechert potentials are given by

$$\phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r} - \vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r) \cdot (\vec{r} - \vec{w}(t_r))}$$
$$\vec{A}(\vec{r}, t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t)$$

In our case $|\vec{r} - \vec{w}(t)| = \sqrt{z^2 + a^2}$ and $\vec{v}(t) \cdot (\vec{r} - \vec{w}(t)) = 0$ for any t so we get

$$\phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0\sqrt{z^2 + a^2}}$$
$$\vec{A}(\vec{r}, t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r}, t) \;=\; \frac{\mu_0 q}{4\pi\sqrt{z^2 + a^2}}[-\sin(t - t_r)\hat{e}_1 + \cos(t - t_r)\hat{e}_2]$$

where $t_r = \frac{1}{c}\sqrt{z^2 + a^2}$ is the retarded time.

(b). By symmetry, the time-averaged electric field is collinear to the z axis so it is sufficient to find $E_z(z, 0, 0)$

$$\langle E_z(z, 0, 0)\rangle \;=\; -\langle\frac{\partial}{\partial z}\phi(z, 0, 0)\rangle - \langle\partial A_z\partial t\rangle \;=\; -\frac{\partial}{\partial z}\phi(z, 0, 0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2 + a^2)^{3/2}}$$

so $\langle\vec{E}(z, 0, 0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}\hat{e}_3$.

Problem 4.

A particle of charge q and mass m moves through an empty space with the velocity $\vec{v} = v\hat{e}_1$ ($v \ll c$). At time $t = 0$, the uniform magnetic field $\vec{B} = B\hat{e}_3$ is switched on. How long it will take the particle to lose half of its kinetic energy? (Assume that the magnetic field is sufficiently weak so that the particle loses half of its energy after many revolutions).

Solution

In the uniform magnetic field, the particle moves around the circle with the radius $R = \frac{v}{\omega_B}$ where $\omega_B = \frac{qB}{m}$ is the cyclotron frequency. Our particle will radiate so v (and R) will slowly decrease with time. The radiated power is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2\omega_B^2 v^2}{6\pi c} = \frac{\mu_0 q^2\omega_B^2}{3\pi mc}\,\frac{mv^2}{2}$$

so we have the differential equation

$$P = -\frac{dE_{\rm kin}(t)}{dt} = -\frac{\mu_0 q^2\omega_B^2}{3\pi mc}E_{\rm kin}(t)$$

The solution is

$$E_{\rm kin}(t) = E_0 e^{-\frac{\mu_0 q^2\omega_B^2}{3\pi mc}t}$$

so the particle will lose half of its kinetic energy after time

$$t = \frac{3\pi mc}{\mu_0 q^2\omega_B^2}\ln 2 = \frac{3\pi m^3 c}{\mu_0 q^4 B^2}\ln 2$$

Problem 5.

In a certain frame K the electric field $\vec{E}$ and the magnetic filed $\vec{B}$ are orthogonal. Is there a frame where the field is

(a) purely electric or (b) purely magnetic,

and with what velocity should that frame(s) move with respect to K?

Solution

In the frame K the second Lorentz invariant $\vec{E}\cdot\vec{B}=0$ while the first invariant E^2-B^2 is positive when $|\vec{E}|>|\vec{B}|$ and negative for $|\vec{E}|<|\vec{B}|$ so nothing forbids to have $\vec{B}'=0$ in the former case and $\vec{E}'=0$ in the latter. To get the velocity of the relevant boost, consider the Lorentz transformations of the electric and magnetic fields

$$\vec{E}' = \gamma(\vec{E}+\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' = \gamma(\vec{B}-\vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

In the first case (when $E>B$) we want to have $B'=0$ so

$$(\vec{B}-\vec{\beta}\times\vec{E}) \;=\; \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

Multiplying both sides by $\vec{\beta}$ we get

$$\vec{\beta}\cdot(\vec{B}-\vec{\beta}\times\vec{E}) \;=\; \frac{\gamma}{\gamma+1}\beta^2(\vec{\beta}\cdot\vec{B}) \;\Rightarrow\; \vec{\beta}\cdot\vec{B}=0 \;\Leftrightarrow\; \vec{v}\perp\vec{B}$$

so $\vec{B}=-\vec{\beta}\times\vec{E}$. The simplest choice is to take $\vec{v}$ orthogonal to both $\vec{E}$ and $\vec{B}$, then

$$v=c\frac{B}{E},\quad \vec{v}\perp\vec{E},\vec{B}$$

Similarly, in the second case we can take $v=c\frac{E}{B},\ \ \vec{v}\perp\vec{E},\vec{B}$ and the resulting $\vec{E}'$ will vanish.

(In the SI units $v=c^2\frac{B}{E}$ and $v=\frac{E}{B}$ for the first and the second case, respectively).

Problem 6.

A π^+ meson with mass 139.6 MeV decays into μ^+-meson with mass 105.7 MeV and massless ν_μ neutrino. What is the velocity (in units of c) of μ^+-meson in the c.m. frame of μ^+ and ν_μ?

Solution

Conservation of energy in the c.m. frame reads ($c=1$)

$$m_\pi \;=\; \sqrt{m_\mu^2+\vec{p}^2}+|\vec{p}|$$

where $\vec{p}$ and $-\vec{p}$ are the momenta of μ-meson and neutrino. We get

$$|\vec{p}| \;=\; \frac{m_\pi^2-m_\mu^2}{2m_\pi} \;=\; \frac{m_\mu v}{\sqrt{1-v^2}} \quad\Rightarrow\quad \frac{v}{\sqrt{1-v^2}} \;=\; \frac{m_\pi^2-m_\mu^2}{2m_\pi m_\mu} \;\equiv\; r \;\simeq\; 0.281$$

so $v \;=\; \sqrt{\frac{r}{1+r}} \;\simeq\; 0.46c$