Problem 1.

Find the flux of the electromagnetic energy (Poyning vector) far away from a uniformly charged disc, with charge Q and radius R, spinning at the angular velocity ω. Does the disc radiate?

Solution

First, the electric charge density does not depend on time, so the electric field is the same as for non-rotating disc. It does not depend on time and the asymptotics at large r is $\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$

Also, we have steady current $\vec{J} = \vec{\omega} r\sigma$ so it is a case described by magnetostatics. The magnitude of magnetic dipole of the disc is

$$m = \sum \Delta I(s)\pi s^2 = \int_0^R ds\ I(s)\pi s^2 = \int_0^R ds(\sigma\omega s)\pi s^2 = \frac{\pi}{4}\omega R^4$$

and the direction is $\hat{z}$ so

$$\vec{m} = \frac{\pi}{4}\omega R^4 \hat{z} = \frac{1}{4} Q\omega R^2 \hat{z}$$

where $Q = \pi R^2 \sigma$ is the charge of the disc.

The fields radiation zone $r \gg R$ are

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, \qquad \vec{B} = \frac{\mu_0}{4\pi r^3}[3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}]$$

and therefore

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{Q}{16\pi^2}\frac{\vec{m}\times\hat{r}}{r^5}$$

In spherical coordinates and therefore

$$\vec{S} = \frac{Q^2 R^2}{64\pi^2}\frac{\sin\theta}{r^5}\hat{\phi}$$

Since $\vec{S} \sim \frac{1}{r^5}$ the disc does not radiate.

Problem 2

A circularly polarized electromagnetic plane wave traveling in the vacuum in upward z direction is incident on an $z > 0$ half-space filled with a dielectric with permittivity $\epsilon > \epsilon_0$ and susceptibility μ_0. Find the charge and current densities induced on the boundary.

Reminder: correct sign of $\vec{K}$ on the boundary between media "1" and media "2" is

$$\vec{K} = \hat{n}_{1\to 2} \times (\vec{H}^{(2)} - \vec{H}^{(1)})$$

where $n_{1\to 2}$ is a unit vector normal to the surface and pointing from media "1" to media "2".

Solution

The incident wave has the form

$$\vec{E} \;=\; E_0(\hat{e}_x + i\hat{e}_y)e^{-i\omega t+ikz}, \quad \vec{B} \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; \frac{\hat{e}_z}{c}\times\vec{E} \;=\; -\,i(\hat{e}_x+i\hat{e}_y)\frac{E_0}{c}e^{-i\omega t+ikz}$$

Since in the case of normal incidence $\frac{E_R}{E_0} = \frac{1-n}{n+1}$ and $\frac{E_T}{E_0} = \frac{2}{n+1}$ for any polarization (see Eq. (7.4.27) from Chapter 7), the reflected wave is

$$\vec{E} \;=\; \frac{1-n}{n+1}E_0(\hat{e}_x+i\hat{e}_y)e^{-i\omega t-ikz},$$
$$\vec{B} \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; -\frac{\hat{e}_z}{c}\times\vec{E} \;=\; -\frac{\hat{e}_z}{c}\times(\hat{e}_x+i\hat{e}_y)E_0\frac{1-n}{n+1}\;e^{-i\omega t-ikz} \;=\; \frac{i}{c}(\hat{e}_x+i\hat{e}_y)\frac{1-n}{n+1}E_0e^{-i\omega t-ikz}$$

and the transmitted one

$$\vec{E} \;=\; \frac{2}{n+1}E_0(\hat{e}_x+i\hat{e}_y)e^{-i\omega t+i\tilde{k}z},$$
$$\vec{B} \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; \frac{\hat{e}_z}{c}\times\vec{E} \;=\; \frac{\hat{e}_z}{c}\times(\hat{e}_x+i\hat{e}_y)E_0\frac{2}{n+1}\;e^{-i\omega t+i\tilde{k}z} \;=\; -\frac{i}{c}(\hat{e}_x+i\hat{e}_y)\frac{2}{n+1}E_0e^{-i\omega t+i\tilde{k}z}$$

where $\tilde{k} = kn$. These are the electric and magnetic fields at $z > 0$. At $z < 0$ we have a sum of incident and reflected wave with the electric and magnetic fields

$$\vec{E} \;=\; \Big(e^{ikz}+\frac{1-n}{n+1}e^{-ikz}\Big)E_0(\hat{e}_x+i\hat{e}_y)e^{-i\omega t},$$
$$\vec{B} \;=\; -\frac{i}{c}(\hat{e}_x+i\hat{e}_y)\Big(e^{ikz}+\frac{n-1}{n+1}e^{-ikz}\Big)E_0e^{-i\omega t}$$

Quick check: boundary condition for the electric field is $E_{\parallel}^{\rm above} = E_{\parallel}^{\rm below}$ (at $z = 0$) is satisfied.

Now, σ and K can be found from the remaining boundary conditions

$$\epsilon_1\vec{E}_{\perp}^{\rm above} - \epsilon_2\vec{E}_{\perp}^{\rm below} = \sigma, \quad \vec{K} = \hat{e}_3\times(\vec{H}^{\rm above}-\vec{H}^{\rm below})$$

We get $\sigma = 0$ and

$$\vec{K} \;=\; i\frac{E_0}{\mu_0 c}\hat{e}_z\times(\hat{e}_x+i\hat{e}_y)\Big[-\frac{2}{n+1}+\Big(1+\frac{n-1}{n+1}\Big)\Big]e^{-i\omega t} \;=\; i(\hat{e}_x+i\hat{e}_y)\frac{E_0}{\mu_0 c}\frac{2n-2}{n+1}e^{-i\omega t}$$

. Taking the real part we get

$$K_x \;=\; \frac{2}{\mu_0 c}E_0\frac{2n-2}{n+1}\sin\omega t, \quad K_y \;=\; -\frac{E_0}{\mu_0 c}\frac{2n-2}{n+1}\cos\omega t$$

Quick check: at $n = 1$ no current, as $n \to \infty$

$$K_x \;=\; \frac{4E_0}{\mu_0 c}\sin\omega t, K_y \;=\; -\frac{4E_0}{\mu_0 c}\cos\omega t$$

as for the reflection from a perfect conductor.

Problem 3.

A wire loop of radius a and resistance R lies in the XY plane. There is a uniform magnetic field $\vec{B} = B\hat{z}$ filling the whole space.

1. What total charge passes a given point in the loop when it is rotated by 90^o around the x axis?

2. Calculate the average power radiated by the loop rotating around X axis with a constant angular frequency ω.

Solution

1.

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

2.

The induced current is

$$I = \frac{1}{R}\frac{d\Phi}{dt} = \frac{B}{R}\pi a^2 \omega \sin\theta(t)$$

so the magnetic moment takes the form

$$\vec{m} = I\pi a^2(\hat{e}_x \cos\theta - \hat{e}_y \sin\theta) = \frac{B}{R}\pi^2 a^4 \omega \sin\theta(\hat{e}_x \cos\theta - \hat{e}_y \sin\theta) = \frac{B}{2R}\pi^2 a^4 \omega\ [\hat{e}_x \sin 2\theta + \hat{e}_y \cos 2\theta - \hat{e}_y]$$

and therefore ($\dot{\theta} = \omega$)

$$\ddot{\vec{m}} = -\frac{2B}{R}\pi^2 a^4 \omega^3 [\hat{e}_x \sin 2\theta + \hat{e}_y \cos 2\theta] \quad \Rightarrow \quad \ddot{\vec{m}}^2 = \frac{4B^2}{R^2}\pi^4 a^8 \omega^6$$

The power is

$$P_{\text{SI}} = \frac{\mu_0}{4\pi} \times \frac{\ddot{m}^2}{3c^3} = \frac{\mu_0}{4\pi} \times \frac{16B^2}{3R^2c^3}\pi^4 a^8 \omega^6$$

Problem 4.

Consider the cylindrical wave guide of radius R closed at one end ($\equiv$ resonant cylindrical cavity opened at one end).

(a) Do we have cutoff frequencies or allowed resonant frequencies in this case?

(b) Find the lowest of them for the **TE** modes (the roots of Bessel functions may be found in *Jackson*).

Solution.

A general form of a running TE wave is

$$H_z \;=\; \psi(s,\phi)(ae^{ikz} + be^{-ikz})e^{-i\omega t}$$

where a and b are arbitrary coefficients and

$$\psi(s,\phi) \;=\; AJ_{mn}(\gamma_{mn}s)e^{\pm im\phi}, \qquad \gamma_{mn} \;\equiv\; \frac{x'_{mn}}{R}$$

where the frequency is

$$\omega \;=\; c\sqrt{\gamma_{mn}^2 + k^2}$$

The corresponding transverse electric field has the form

$$\vec{E}_T \;=\; -\frac{i\mu\omega}{\gamma^2}\hat{z}\times\vec{\nabla}_T H_z = -\frac{i\omega\mu}{\gamma^2}(ae^{ikz} + be^{-ikz})\,\hat{z}\times\vec{\nabla}_T\psi$$

In the case of wave guide closed at one end (say, at $z=0$) we have $\vec{E}_T\Big|_{z=0} = 0 \;\;\Rightarrow\;\; a = -b$ so we get

$$H_z \;=\; a\psi(s,\phi)\sin(kz)e^{-i\omega t}$$

where k is arbitrary.

Thus, we have cutoff frequencies in this case and the lowest cutoff frequency is

$$\omega_{11} \;=\; \frac{1.811}{R}c$$

Problem 5.

A photon of energy $E_{\rm th}$ collides at angle $\theta = 40°$ with another photon of energy $E = 2\frac{\rm GeV}{c^2}$. Find the minimum value of $E_{\rm th}$ permitting formation of a proton-antiproton pair. The mass of a proton (and antiproton) is 938 $\frac{\rm MeV}{c^2}$. (Reminder: 1 Gev = 1000 MeV).

Solution

$$E_{\rm th} \;=\; \frac{2m_p^2}{E(1-\cos\theta)} \;=\; 3.76{\rm GeV}$$

Problem 6.

In a certain frame K the electric field $\vec{E}$ and the magnetic filed $\vec{B}$ are identical (in Gauss units). Is there a frame moving in the direction orthogonal to $\vec{E} = \vec{B}$ where the the angle between the fields $\vec{E}'$ and $\vec{B}'$ is

(a) 60°, (b) 90°, c) 120°

and with what velocity should that frame(s) move with respect to K?

Solution

First, from the Lorentz invariant $\vec{E}\cdot\vec{B} = E^2 > 0$ we see that the angles 120° and 90° are not possible (since $\vec{E}'\cdot\vec{B}' < 0$ for 120° and $\vec{E}'\cdot\vec{B}' = 0$ for 90°. The angle 60° is possible. Let us find the relevant boost velocity. After the Lorentz boost we have

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

For the boost in the direction orthogonal to $\vec{E} = \vec{B}$ the last terms in the above equation drop so we have

$$\begin{aligned}\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) &= \gamma(E\hat{e}_3 - \beta E\hat{e}_2)\\ \vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) &= \gamma(E\hat{e}_3 + \beta E\hat{e}_2)\end{aligned}$$

where we've chosen $\vec{E} = \vec{B} \parallel \hat{e}_3$ and $\beta \parallel \hat{e}_1$. The angle 60° between $\vec{E}'$ and $\vec{B}'$ means that

$$\frac{\vec{E}'\cdot\vec{B}'}{|\vec{E}'||\vec{B}'|} = \cos 60^\circ = \frac{1}{2} \quad\Leftrightarrow\quad \frac{1-\beta^2}{1+\beta^2} = \frac{1}{2} \quad\Rightarrow\quad v = \frac{c}{\sqrt{3}}$$