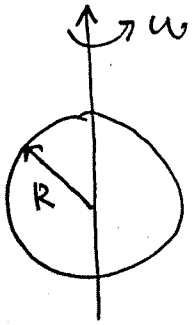


# HW 3 solution



Fields:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \theta(r-R) = \frac{e}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \theta(r-R)$$

$$\vec{B}_{\text{inside}} = \frac{2}{3} \mu_0 \zeta R \vec{\omega}$$

Boutside  $\approx$  mag. field of a pure <sup>dipole</sup>

From Griffiths:  $\vec{A}(r, \theta, \varphi) = \frac{\mu_0 R^4}{3} \omega \zeta \frac{\sin\theta}{r^2} \hat{\varphi} =$

$$= \frac{\mu_0}{4\pi} \left( \frac{4\pi}{3} R^4 \omega \zeta \right) \frac{\sin\theta}{r^2} \hat{\varphi}$$

$$\Rightarrow \vec{B}(r, \theta, \varphi) = \frac{\mu_0}{4\pi} \left( \frac{4\pi}{3} R^4 \omega \zeta \right) \frac{2\cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} \quad \zeta = \frac{e}{4\pi R^2}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 R^4}{3} \omega \zeta \frac{2\cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} \theta(r-R) + \frac{2}{3} \mu_0 \zeta R \omega \hat{e}_3 \theta(R-r)$$

Energy:

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{e^2 \theta(r-R)}{32\pi^2 \epsilon_0 r^4} \left\{ 1 + \mu_0 \epsilon_0 \frac{\omega^2 R^4}{9 r^2} (1 + 3\cos^2\theta) \right\} + \frac{\mu_0}{72\pi^2} \frac{e^2 \omega^2}{R^2} \theta(R-r)$$

$$\int d^3x u = \frac{4}{3} \pi R^3 \frac{\mu_0}{72\pi^2} \frac{e^2 \omega^2}{R^2} + 2\pi \int_R^\infty r^2 dr \int_0^\pi d\theta \frac{e^2 \sin\theta}{32\pi^2 \epsilon_0} \left[ \frac{1}{r^4} + \frac{\mu_0 \epsilon_0 \omega^2 R^4}{9 r^6} (1 + 3\cos^2\theta) \right]$$

$$= \frac{\mu_0}{54\pi} e^2 \omega^2 R + \frac{e^2}{8\pi\epsilon_0} \left( \frac{1}{R} + \frac{2}{27} \mu_0 \epsilon_0 \omega^2 R \right)$$

$$= \frac{e^2}{8\pi\epsilon_0 R} \left( 1 + \left[ \frac{8}{54} + \frac{2}{27} \right] \mu_0 \epsilon_0 \omega^2 R^2 \right) = W_{\text{e.m.}} = \frac{e^2}{8\pi\epsilon_0 R} \left( 1 + \frac{2}{9} \frac{\omega^2 R^2}{c^2} \right)$$

Angular momentum

$$\vec{L} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$$

$$\vec{E} \times \vec{B} = \frac{e^2 \mu_0 R^2 \omega}{48 \pi^2 \epsilon_0 r^5} \hat{r} \times (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \theta(r-R)$$

$$\epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \frac{e^2 \mu_0 R^2 \omega}{48 \pi^2 r^4} \hat{r} \times (\hat{r} \times \hat{\theta}) \sin \theta = - \frac{e^2 \mu_0 R^2 \omega}{48 \pi^2} \frac{\sin \theta}{r^4} \hat{\theta}$$

$$\int d^3x \vec{L} = - \frac{\mu_0 \omega e^2 R^2}{48 \pi^2} \int_R^\infty r^2 dr \int_0^\pi d\theta \sin^2 \theta \int_0^{2\pi} d\varphi \frac{\hat{\theta}}{r^4}$$

By symmetry  $\int d\varphi \hat{\theta} \uparrow \uparrow \hat{e}_3 \Rightarrow \hat{\theta}_z = -\sin \theta$

$$\Rightarrow \int d^3x L_z = + \frac{\mu_0 \omega e^2 R^2}{48 \pi^2} \frac{2\pi}{R} \int_0^\pi d\theta \sin^3 \theta = \frac{\mu_0 \omega R e^2}{18\pi} \Rightarrow \vec{L} = \frac{\mu_0 \omega R}{18\pi} e^2 \hat{z}$$

Classical model?

$$m_e c^2 = \frac{e^2}{8\pi \epsilon_0 R}$$

$$\frac{\mu_0 \omega R}{18\pi} e^2 = \frac{\hbar}{2} \Rightarrow \mu_0 \omega R = \frac{9\pi \hbar}{e^2} \Rightarrow \mu_0 \epsilon_0 \omega R = \frac{9\pi \hbar \epsilon_0}{e^2}$$

$$\Rightarrow \frac{\omega R}{c} = \frac{v}{c} = \frac{9\pi \hbar c \epsilon_0}{e^2} = \frac{9}{4} \frac{1}{e^2 / 4\pi \hbar c \epsilon_0} = \frac{9}{4} \cdot 137 \gg 1$$

$$\frac{e^2}{4\pi \hbar c \epsilon_0} = \text{"fine structure constant"} \approx \frac{1}{137}$$