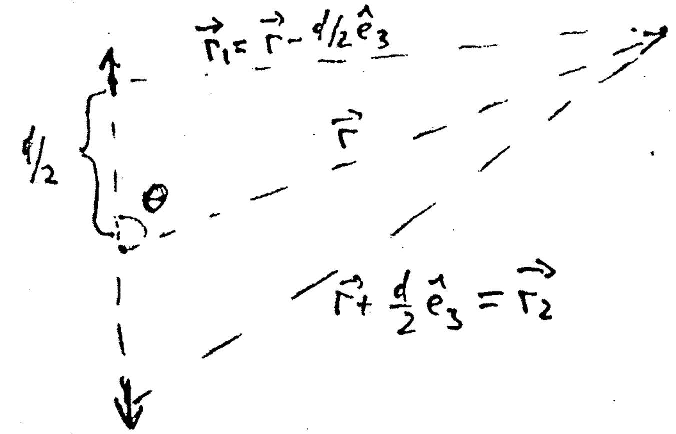
**Problem 9.13** As a model for electric quadrupole radiation, consider two oppositely oriented oscillating electric dipoles, separated by a distance $d$, as shown in Fig. 9.12. (Use the results of Section 9.1.2 for the potentials of each dipole, but note that they are *not* located at the origin.) Keeping only terms of first order in $d$,

(a) Find the scalar and vector potentials.

(b) Find the electric and magnetic fields.

(c) Find the Poynting vector and the power radiated. Sketch the intensity profile as a function of $\theta$.




For 1 dipole

$$\phi(\vec{r},t) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r}\, e^{ikr}\,(e^{-i\omega t})$$

$$\vec{H}(\vec{r},t) = \frac{ck^2}{4\pi}\,\hat{r}\times\vec{p}\,\frac{e^{ikr}}{r}\,(e^{-i\omega t})$$

$$A(\vec{r},t) = -\frac{i\omega\mu_0}{4\pi r}\,\vec{p}\,e^{ikr}\,(e^{-i\omega t})$$

For 2 dipoles

$$\phi = \frac{\hat{r}_1\cdot\vec{p}}{4\pi\epsilon_0 r_1}\, e^{ikr_1} - \frac{\hat{r}_2\cdot\vec{p}}{4\pi\epsilon_0 r_2}\, e^{ikr_2}$$

$$kr_1 \approx k|\vec{r} - d/2\,\hat{e}_3| = kr - \frac{kd}{2}\,\hat{r}\cdot\hat{e}_3 \Rightarrow$$

$$\Rightarrow e^{ikr_1} \approx e^{ikr}\left(1 - \frac{ikd}{2}\cos\theta\right)$$

$$e^{ikr_1} - e^{ikr_2} = -ikd\cos\theta\, e^{ikr}$$

Similarly, $\exp(ikr_2) = e^{ikr}\left(1 + \frac{ikd}{2}\cos\theta\right)$

$$\Rightarrow \phi \approx \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r}\, e^{ikr}\left(1 - \frac{ikd}{2}\cos\theta - 1 - \frac{ikd}{2}\cos\theta\right) + O\left(\frac{1}{r^2}\right)$$

$$\phi(\vec{r}) = -\frac{ikdp}{4\pi\epsilon_0}\,\frac{e^{ikr}}{r}\cos^2\theta$$

$$A(\vec{r}) \simeq -\frac{i\omega\mu_0}{4\pi r}\vec{p}\,(e^{ikr_1}-e^{ikr_2}) = -\frac{\mu_0 ck^2 d}{4\pi r}e^{ikr}\cos\theta\,\vec{p}$$

$$\vec{H}(\vec{r}) \simeq \frac{ck^2}{4\pi r}\hat{r}\times\vec{p}\,(e^{ikr_1}-e^{ikr_2}) = \frac{ck^2 p}{4\pi r}\underbrace{\hat{r}\times\hat{e}_3}_{-\hat{e}_\varphi\sin\theta}(-ikd\cos\theta)\cdot$$

$$\cdot e^{ikr} = i\frac{k^3 dc}{4\pi}\sin\theta\cos\theta\,\hat{e}_\varphi = H(\vec{r})$$

Check: $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{A} = -\frac{\mu_0 ck^2 d}{4\pi r}e^{ikr}\cos\theta\,\vec{p}$$

In cylindrical coordinates $\vec{p} \equiv p\hat{e}_3$ $\quad r = \sqrt{s^2+z^2}$

$$\vec{\nabla}\times\vec{A} = -\hat{e}_\varphi\frac{\partial A_3}{\partial s}$$

$$\frac{\partial}{\partial s}\frac{e^{ikr}}{r}\cos\theta = \frac{\partial}{\partial s}z\frac{e^{ik\sqrt{z^2+s^2}}}{z^2+s^2} \simeq ik\frac{sz}{(z^2+s^2)^{3/2}}e^{ik\sqrt{z^2+s^2}}$$

$$= ik\sin\theta\cos\theta\frac{e^{ikr}}{r}$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = i\mu_0\frac{ck^3 d}{4\pi r}\sin\theta\cos\theta\frac{e^{ikr}}{r}e_{\hat{\varphi}} \quad - \text{ ok}$$

$$\vec{E}(\vec{r}) = \mu_0\hat{H}\times\hat{n} = \frac{ik^3 d}{4\pi\epsilon_0}\sin\theta\cos\theta\,\hat{e}_\varphi\times\hat{r} = \frac{ik^3 d}{4\pi\epsilon_0}\sin\theta\cos\theta\,\hat{e}_\theta$$

Power

$$\frac{dP}{d\Omega} = \frac{Z_0}{2}r^2|H|^2 = \frac{\mu_0 c}{2}\left(\frac{k^3 dc}{4\pi}\right)^2\sin^2\theta\cos^2\theta$$

$$P = \int d\Omega\frac{dP}{d\Omega} = 2\pi\int_0^\pi \sin\theta\,d\theta\quad\frac{\mu_0 c^3 k^6 d^2}{32\pi^2}\sin^2\theta\cos^2\theta$$

$$= \frac{\mu_0 c^3 k^6 d^2}{16\pi}\int_{-1}^{1}dt\,t^2(1-t^2) = \frac{\mu_0 c^3 k^6 d^2}{60\pi} = P$$