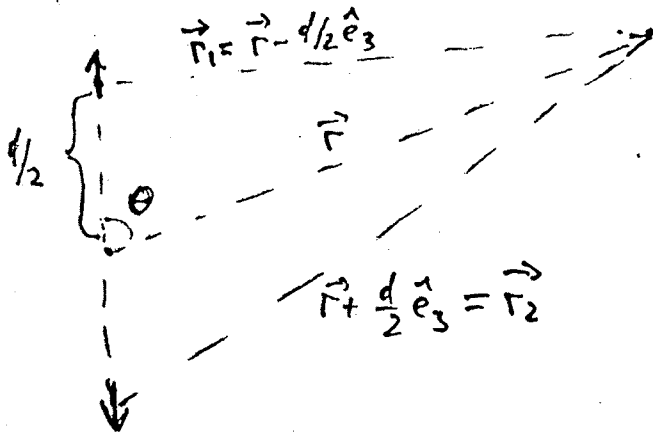


- Problem 9.13** As a model for electric quadrupole radiation, consider two oppositely oriented oscillating electric dipoles, separated by a distance d , as shown in Fig. 9.12. (Use the results of Section 9.1.2 for the potentials of each dipole, but note that they are *not* located at the origin.) Keeping only terms of first order in d ,
- Find the scalar and vector potentials.
 - Find the electric and magnetic fields.
 - Find the Poynting vector and the power radiated. Sketch the intensity profile as a function of θ .



For 1 dipole

$$\phi(\vec{r}, t) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r} e^{ikr} (e^{-i\omega t})$$

$$\vec{H}(\vec{r}, t) = \frac{ck^2}{4\pi} \hat{r} \times \vec{p} \frac{e^{ikr}}{r} (e^{-i\omega t})$$

$$\vec{A}(\vec{r}, t) = -\frac{i\omega\mu_0}{4\pi r} \vec{p} e^{ikr} (e^{-i\omega t})$$

For 2 dipoles

$$\phi = \frac{\hat{r}_1 \cdot \vec{p}}{4\pi\epsilon_0 r_1} e^{ikr_1} - \frac{\hat{r}_2 \cdot \vec{p}}{4\pi\epsilon_0 r_2} e^{ikr_2}$$

$$kr_1 = k|\vec{r} - \frac{d}{2}\hat{e}_3| = kr - \frac{kd}{2} \hat{r} \cdot \hat{e}_3 \Rightarrow e^{ikr_1} = e^{ikr} (1 - \frac{ikd}{2} \cos\theta)$$

$$\Rightarrow e^{ikr_1} \approx e^{ikr} (1 - \frac{ikd}{2} \cos\theta)$$

$$e^{ikr_1} - e^{ikr_2} = -ikd \cos\theta e^{ikr}$$

$$\text{Similarly, } \exp(ikr_2) = e^{ikr} (1 + \frac{ikd}{2} \cos\theta)$$

$$\Rightarrow \phi = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r} e^{ikr} (1 - \frac{ikd}{2} \cos\theta - 1 - \frac{ikd}{2} \cos\theta) + o(\frac{1}{r_2})$$

$$\phi(\vec{r}) = -\frac{ikd p}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \cos^2\theta$$

$$A(\vec{r}) \approx -\frac{i\omega\mu_0}{4\pi r} \vec{p} (e^{ikr_1} - e^{ikr_2}) = -\frac{\mu_0 c k^2 d}{4\pi r} e^{ikr} \cos\theta \vec{p}$$

$$\vec{H}(\vec{r}) = \frac{ck^2}{4\pi r} \hat{r} \times \vec{p} (e^{ikr_1} - e^{ikr_2}) = \frac{ck^2 p}{4\pi r} \begin{matrix} \hat{r} \times \hat{e}_3 \\ -\hat{e}_\varphi \sin\theta \end{matrix} (-ikd \cos\theta) \cdot e^{ikr} = i \frac{k^3 d c}{4\pi} \sin\theta \cos\theta \hat{e}_\varphi = H(\vec{r})$$

Check: $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{A} = -\frac{\mu_0 c k^2 d}{4\pi r} e^{ikr} \cos\theta \vec{p}$$

In cylindrical coordinates $\vec{p} = p \hat{e}_3$ $r = \sqrt{z^2 + s^2}$

$$\vec{\nabla} \times \vec{A} = -\hat{e}_\varphi \frac{\partial A_3}{\partial s}$$

$$\frac{\partial}{\partial s} \frac{e^{ikr}}{r} \cos\theta = \frac{\partial}{\partial s} z \frac{e^{ik\sqrt{z^2+s^2}}}{z^2+s^2} \approx ik \frac{sz}{(z^2+s^2)^{3/2}} e^{ik\sqrt{z^2+s^2}} = ik \sin\theta \cos\theta \frac{e^{ikr}}{r}$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = i\mu_0 \frac{ck^3 d}{4\pi r} \sin\theta \cos\theta \frac{e^{ikr}}{r} \hat{e}_\varphi - 0k$$

$$\vec{E}(\vec{r}) = \mu_0 \hat{H} \times \hat{n} = \frac{ik^3 d}{4\pi \epsilon_0} \sin\theta \cos\theta \hat{e}_\varphi \times \hat{r} = \frac{ik^3 d}{4\pi \epsilon_0} \sin\theta \cos\theta \hat{e}_\theta$$

Power

$$\frac{dP}{d\Omega} = \frac{z_0}{2} r^2 |H|^2 = \frac{\mu_0 c}{2} \left(\frac{k^3 d c}{4\pi}\right)^2 \sin^2\theta \cos^2\theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = 2\pi \int_0^\pi \sin\theta d\theta \frac{\mu_0 c^3 k^6 d^2}{32\pi^2} \sin^2\theta \cos^2\theta = \frac{\mu_0 c^3 k^6 d^2}{16\pi} \int_{-1}^1 dt t^2 (1-t^2) = \frac{\mu_0 c^3 k^6 d^2}{60\pi} = P$$