804 Midterm (20 points) March 15 2012, 10:45 ÷ 12:15.

Problem 1.

A wire loop of radius a and resistance R lies in the XY plane. There is a uniform magnetic field $\vec{B} = B\hat{z}$ filling the whole space. What total charge passes a given point in the loop when it is rotated by 90^o around the x axis?

Solution

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

Problem 2.

Find the reflection coefficient for the circularly polarized electromagnetic wave incident on a plane between two linear media at Brewster's angle. (For simplicity, take $\mu' = \mu$).

Solution

The circularly polarized wave have the form

$$\vec{E} = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)E_0^i e^{ik(z\cos\theta_i + x\sin\theta_i)}$$

(we've assumed that $\vec{k}$ lies in the XZ plane).

Let us consider the components $E_0^i\hat{e}_1$ (electric field in the plane of incidence) and $iE_0^i\hat{e}_2$ (electric field normal to the plane of incidence) separately.

At $\theta_i = \theta_B$ we have no reflected wave for the component $E_0^i\hat{e}_1$ (in the plane of incidence). For $iE_0^i\hat{e}_2$ (normal to the plane of incidence) we have

$$E_0^r = \left(\frac{i}{\sqrt{2}}E_0^i\right)\frac{\sin(\theta_T - \theta_i)}{\sin(\theta_T + \theta_i)}$$

Since at $\theta_i = \theta_B$ we have $\sin\theta_B = \cos\theta_T = n'/\sqrt{n^2 + n'^2}$ and $\cos\theta_B = \sin\theta_T = n/\sqrt{n^2 + n'^2}$, we obtain

$$E_0^r = \frac{i}{\sqrt{2}}E_0^i\frac{\sin(\theta_T - \theta_B)}{\sin(\theta_T + \theta_B)} = \frac{i}{\sqrt{2}}E_0^i\frac{n^2 - n'^2}{n^2 + n'^2}$$

Therefore, the intensity of the reflected wave is proportional to $\frac{1}{2}\left(E_0^i\frac{n^2 - n'^2}{n^2 + n'^2}\right)^2$ and the reflection coefficient is

$$R = \frac{(n^2 - n'^2)^2}{2(n^2 + n'^2)^2}$$

Problem 3.

A $\mathrm{TE}_{1,1}$ wave propagates along the rectangular wave guide made from a perfect conductor (for simplicity take $a = b$).
(a) Find the (surface) currents on the surface of the guide.
(b) Is there a total current flow along the wave guide?

Solution
The field for $\mathrm{TE}_{1,1}$ wave have the form (here $\gamma^2 = \gamma_{1,1}^2 = \frac{2\pi^2}{a^2}$)

$$
\begin{aligned}
H_z &= H_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a} e^{ikz-i\omega t} \\
\vec{H}_T &= \frac{ik}{\gamma^2}\vec{\nabla}_T H_z \;=\; -i\frac{\pi k}{a\gamma^2}\Big(\hat{e}_1 \sin\frac{\pi x}{a}\cos\frac{\pi y}{a} + \hat{e}_2\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\Big)e^{ikz-i\omega t} \\
\vec{E}_T &= = -\frac{i\mu\omega}{\gamma^2}\hat{e}_3\times\vec{\nabla}_T H_z \;=\; i\frac{\pi\mu_0\omega}{a\gamma^2}\Big(\hat{e}_1\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} - \hat{e}_2\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big)e^{ikz-i\omega t}
\end{aligned}
$$

Let us calculate the surface current at the bottom $y = 0$. From the boundary condition $\hat{n}\times\vec{H} = \vec{K}$ we get (here $\vec{n} = -\hat{e}_2$)

$$
\vec{K}\Big|_{y=0} = -\hat{e}_1 H_0\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{y=0} - \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{y=0} \;=\; -\hat{e}_1 H_0\cos\frac{\pi x}{a} \;-\; \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi x}{a}
$$

At the top $y = a$ one gets ($\vec{n} = \hat{e}_2$)

$$
\vec{K}\Big|_{y=a} = \hat{e}_1 H_0\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{y=a} + \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{y=a} \;=\; -\hat{e}_1 H_0\cos\frac{\pi x}{a} \;-\; \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi x}{a}
$$

Similarly, for the two sides $x = 0$ and $x = a$ we obtain

$$
\begin{aligned}
\vec{K}\Big|_{x=0} &= \hat{e}_2 H_0\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{x=0} + \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\Big|_{x=0} \;=\; \hat{e}_2 H_0\cos\frac{\pi y}{a} \;+\; \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi y}{a} \\
\vec{K}\Big|_{x=a} &= -\hat{e}_2 H_0\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{x=a} - \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big|_{x=a} \;=\; \hat{e}_2 H_0\cos\frac{\pi y}{a} \;+\; \hat{e}_3\frac{i\pi k}{a\gamma^2}H_0\sin\frac{\pi y}{a}
\end{aligned}
$$

The total current along the wave guide vanishes:

$$
I_3 \;=\; \int_0^a dx\; K_3(0,x) + \int_0^a dy\; K_3(a,y) + \int_0^a dx\; K_3(a,x) + \int_0^a dy\; K_3(0,y) \;=\; 0.
$$