

Problem 1.

A wire loop of radius a and resistance R lies in the XY plane. There is a uniform magnetic field $\vec{B} = B\hat{z}$ filling the whole space. What total charge passes a given point in the loop when it is rotated by 90° around the x axis?

Solution

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

Problem 2.

Find the reflection coefficient for the circularly polarized electromagnetic wave incident on a plane between two linear media at Brewster's angle. (For simplicity, take $\mu' = \mu$).

Solution

The circularly polarized wave have the form

$$\vec{E} = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)E_0 e^{ik(z \cos \theta_i + x \sin \theta_i)}$$

(we've assumed that \vec{k} lies in the XZ plane).

Let us consider the components $E_0^i \hat{e}_1$ (electric field in the plane of incidence) and $iE_0^i \hat{e}_2$ (electric field normal to the plane of incidence) separately.

At $\theta_i = \theta_B$ we have no reflected wave for the component $E_0^i \hat{e}_1$ (in the plane of incidence). For $iE_0^i \hat{e}_2$ (normal to the plane of incidence) we have

$$E_0^r = \left(\frac{i}{\sqrt{2}}E_0^i\right) \frac{\sin(\theta_T - \theta_i)}{\sin(\theta_T + \theta_i)}$$

Since at $\theta_i = \theta_B$ we have $\sin \theta_B = \cos \theta_T = n'/\sqrt{n^2 + n'^2}$ and $\cos \theta_B = \sin \theta_T = n/\sqrt{n^2 + n'^2}$, we obtain

$$E_0^r = \frac{i}{\sqrt{2}}E_0^i \frac{\sin(\theta_T - \theta_B)}{\sin(\theta_T + \theta_B)} = \frac{i}{\sqrt{2}}E_0^i \frac{n^2 - n'^2}{n^2 + n'^2}$$

Therefore, the intensity of the reflected wave is proportional to $\frac{1}{2}\left(E_0^i \frac{n^2 - n'^2}{n^2 + n'^2}\right)^2$ and the reflection coefficient is

$$R = \frac{(n^2 - n'^2)^2}{2(n^2 + n'^2)^2}$$

Problem 3.

A $\text{TE}_{1,1}$ wave propagates along the rectangular wave guide made from a perfect conductor (for simplicity take $a = b$).

- (a) Find the (surface) currents on the surface of the guide.
 (b) Is there a total current flow along the wave guide?

Solution

The field for $\text{TE}_{1,1}$ wave have the form (here $\gamma^2 = \gamma_{1,1}^2 = \frac{2\pi^2}{a^2}$)

$$\begin{aligned} H_z &= H_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} e^{ikz - i\omega t} \\ \vec{H}_T &= \frac{ik}{\gamma^2} \vec{\nabla}_T H_z = -i \frac{\pi k}{a\gamma^2} \left(\hat{e}_1 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} + \hat{e}_2 \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} \right) e^{ikz - i\omega t} \\ \vec{E}_T &= -\frac{i\mu\omega}{\gamma^2} \hat{e}_3 \times \vec{\nabla}_T H_z = i \frac{\pi\mu_0\omega}{a\gamma^2} \left(\hat{e}_1 \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} - \hat{e}_2 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \right) e^{ikz - i\omega t} \end{aligned}$$

Let us calculate the surface current at the bottom $y = 0$. From the boundary condition $\hat{n} \times \vec{H} = \vec{K}$ we get (here $\vec{n} = -\hat{e}_2$)

$$\vec{K} \Big|_{y=0} = -\hat{e}_1 H_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{y=0} - \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{y=0} = -\hat{e}_1 H_0 \cos \frac{\pi x}{a} - \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi x}{a}$$

At the top $y = a$ one gets ($\vec{n} = \hat{e}_2$)

$$\vec{K} \Big|_{y=a} = \hat{e}_1 H_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{y=a} + \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{y=a} = -\hat{e}_1 H_0 \cos \frac{\pi x}{a} - \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi x}{a}$$

Similarly, for the two sides $x = 0$ and $x = a$ we obtain

$$\begin{aligned} \vec{K} \Big|_{x=0} &= \hat{e}_2 H_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{x=0} + \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} \Big|_{x=0} = \hat{e}_2 H_0 \cos \frac{\pi y}{a} + \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi y}{a} \\ \vec{K} \Big|_{x=a} &= -\hat{e}_2 H_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{x=a} - \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \Big|_{x=a} = \hat{e}_2 H_0 \cos \frac{\pi y}{a} + \hat{e}_3 \frac{i\pi k}{a\gamma^2} H_0 \sin \frac{\pi y}{a} \end{aligned}$$

The total current along the wave guide vanishes:

$$I_3 = \int_0^a dx K_3(0, x) + \int_0^a dy K_3(a, y) + \int_0^a dx K_3(a, x) + \int_0^a dy K_3(0, y) = 0.$$