Problem 1.

A wire loop of radius a and resistance R lies in the XY plane. There is a uniform magnetic field $\vec{B} = B\hat{z}$ filling the whole space. What total charge passes a given point in the loop when it is rotated by 90^o around the x axis?

Solution:

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

Problem 2.

Consider an infinitely long straight wire along the z-axis. Suppose the wire gets a sudden burst of current given by $I(t) = a\delta(t)$ where a is a constant and $\delta(t)$ is the Dirac delta function. Find the electric and magnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$.

Solution:

By symmetry, potential depends only on distance from the wire (and time). Let us calculate potentials at the point $(x,0,0)$. The general formulas are

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{r}',t_{\text{ret}})}{|\vec{r}-\vec{r}'|} ,$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{r}',t_{\text{ret}})}{|\vec{r}-\vec{r}'|} ,$$

so for our case

$$\Phi(\vec{r},t) = 0$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int dz \frac{I(z,t_{\text{ret}})}{|\vec{r}-\vec{r}'|}\hat{e}_z = \frac{\mu_0 a}{4\pi}\int dz \frac{\delta(t-\frac{\sqrt{x^2+z^2}}{c})}{\sqrt{x^2+z^2}}\hat{e}_z = \frac{\mu_0 ac}{2\pi}\frac{\theta(ct-x)}{\sqrt{c^2t^2-x^2}}\hat{e}_z$$

Thus, in cylindrical coordinates

$$\Phi = 0$$

$$A_z(s,t) = \frac{\mu_0 ac}{2\pi}\frac{\theta(ct-s)}{\sqrt{c^2t^2-s^2}}, \qquad A_s = A_\phi = 0$$

where $s \equiv \sqrt{x^2+y^2}$.

The electric and magnetic fields are

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t) - \frac{d\vec{A}(\vec{r},t)}{dt} = \frac{\mu_0 ac^3}{2\pi}\frac{t\theta(ct-s)}{(c^2t^2-s^2)^{\frac{3}{2}}}\hat{e}_z$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = -\frac{dA_z(s,t)}{ds}\hat{e}_\phi = -\frac{\mu_0 ac}{2\pi}\frac{s\theta(ct-s)}{(c^2t^2-s^2)^{\frac{3}{2}}}\hat{e}_\phi$$

Problem 3.

Find the reflection coefficient for the circularly polarized electromagnetic wave incident on a plane between two linear media at Brewster's angle. (For simplicity, take $\mu' = \mu$).

Solution:

The circularly polarized wave have the form

$$\vec{E} = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)E_0^i e^{ik(z\cos\theta_i + x\sin\theta_i)}$$

(we've assumed that $\vec{k}$ lies in the XZ plane).

Let us consider the components $E_0^i\hat{e}_1$ (electric field in the plane of incidence) and $iE_0^i\hat{e}_2$ (electric field normal to the plane of incidence) separately.

At $\theta_i = \theta_B$ we have no reflected wave for the component $E_0^i\hat{e}_1$ (in the plane of incidence). For $iE_0^i\hat{e}_2$) (normal to the plane of incidence) we have

$$E_0^r = \Big(\frac{i}{\sqrt{2}}E_0^i\Big)\frac{\sin(\theta_T - \theta_i)}{\sin(\theta_T + \theta_i)}$$

Since at $\theta_i = \theta_B$ we have $\sin\theta_B = \cos\theta_T = n'/\sqrt{n^2+n'^2}$ and $\cos\theta_B = \sin\theta_T = n/\sqrt{n^2+n'^2}$, we obtain

$$E_0^r = \frac{i}{\sqrt{2}}E_0^i\frac{\sin(\theta_T - \theta_B)}{\sin(\theta_T + \theta_B)} = \frac{i}{\sqrt{2}}E_0^i\frac{n^2-n'^2}{n^2+n'^2}$$

Therefore, the intensity of the reflected wave is proportional to $\frac{1}{2}\Big(E_0^i\frac{n^2-n'^2}{n^2+n'^2}\Big)^2$ and the reflection coefficient is

$$R = \frac{(n^2-n'^2)^2}{2(n^2+n'^2)^2}$$