804 Midterm (16 points). 02/14/20

Problem 1 .

An iron sphere of radius R carries a charge Q and has a uniform magnetization $\vec{M} = M\hat{e}_3$. It is initially at rest. Find

(a) Angular momentum stored in the fields

(b) If the sphere is demagnetized by heating (keeping $\vec{M}$ uniform), by use of Faraday's law find the induced electric field, then find the torque induced by $\vec{E}$ on the sphere, and finally the angular momentum imparted to sphere as M goes to zero.

Hint: The magnetic field outside the sphere is equal to the magnetic field of a pure dipole with $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

Solution

(a)

The electric and magnetic fields (at $r > R$) are:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, \quad \vec{B} = \frac{\mu_0}{4\pi r^3}(3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}) \quad \Rightarrow \quad \vec{E}\times\vec{B} \;=\; \frac{Q\mu_0 m}{16\pi^2\epsilon_0 r^5}\hat{e}_3 \times \vec{r}$$

where $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$.

The angular momentum stored in the fields is

$$\int_R^\infty dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \epsilon_0 \vec{r}\times(\vec{E}\times\vec{B}) = \int_R^\infty dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \; \frac{\mu_0 m Q}{16\pi^2 r^4}(\hat{e}_3 - (\hat{e}_3\cdot\hat{r})\hat{r})$$

By symmetry, the angular momentum $\vec{L}$ is collinear to $\hat{e}_3$ so

$$\vec{L} \;=\; \hat{e}_3 \int_R^\infty dr r^2 \int_0^\pi d\theta \int_0^{2\pi} d\phi \; \frac{\mu_0 m Q}{16\pi^2 r^4}\sin^3\theta \;=\; \frac{\mu_0 Q}{6\pi R}\vec{m} \;=\; \frac{2}{9}\mu_0 Q R^2 M$$

(b)

If $\vec{m} = m(t)\hat{e}_3$ the induced electric field takes the form

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\frac{\mu_0 m(t)\hat{e}_3 \times \hat{r}}{4\pi r^2} = -\frac{\mu_0}{4\pi r^2}\hat{e}_3 \times \hat{r}\frac{dm}{dt} = -\frac{\mu_0}{4\pi r^2}\hat{\phi}\sin\theta\frac{dm}{dt}$$

The torque for the charge dq on the surface is

$$d\vec{\tau} = dq\;\vec{r}\times\vec{E} = dq\frac{\mu_0}{4\pi R}\hat{\theta}\sin\theta\frac{dm}{dt}, \qquad \hat{r}\times\hat{\phi} = -\hat{\theta}$$

Again, from symmetry we know that $\vec{\tau} \parallel \hat{e}_3$ so

$$d\tau_3 \;=\; -\,dq\frac{\mu_0}{4\pi R}\sin^2\theta\frac{dm}{dt}$$

Since the charge is distributed uniformly over the surface of the sphere $dq = \frac{Q}{4\pi R^2} R^2 \sin\theta d\theta d\phi$ and the total torque is

$$\tau_3 \;=\; -\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \;\sin^2\theta \frac{\mu_0 Q}{16\pi^2 R}\frac{dm}{dt} \;=\; -\frac{\mu_0 Q}{6\pi R}\frac{dm}{dt}$$

The angular momentum imparted to sphere is

$$\vec{m} \;=\; \hat{e}_3\int_0^\infty \tau_3(t)dt = -\frac{\mu_0 Q}{6\pi R}\hat{e}_3\int_0^\infty \frac{dm}{dt}dt = \frac{\mu_0 Q}{6\pi R} m\hat{e}_3 = \frac{2}{9}\mu_0 Q R^2 M\hat{e}_3$$

Problem 2 (6 points)

A circular wire of radius a lies in x, y plane with the center at the origin. It carries current

$$I(t) \;=\; I_0\theta(t) \;=\; \begin{cases} 0 & t < 0 \\ I_0 & t \geq 0 \end{cases}$$

Find the magnetic field on the z axis (at $x, y = 0$).

Solution:

Generalization of Bio-Savart law for time-dependent currents has the form (formula (6.56) from *Jackson* or (6.3.23) from Chapter 6 of lecture notes))

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\vec{J}(\vec{x'}, t_{\rm ret})}{R^2} + \frac{\dot{\vec{J}}(\vec{x}', t_{\rm ret})}{cR}\right] \times \hat{e}_R$$

where $t_{\rm ret} = t - \frac{R}{c}$ and $R = |\vec{x} - \vec{x'}|$. Rewriting it for linear currents, one gets

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi}\int \Big[\frac{I(\vec{x'}, t_{\rm ret})}{R^2} + \frac{\dot{I}(\vec{x}', t_{\rm ret})}{cR}\Big]\vec{dl'} \times \hat{e}_R$$

By symmetry, the magnetic field has only z component. Moreover, for a point on z-axis the retarded time is the came for all points on the wire and is equal to $t - \frac{R}{c} = t - \frac{\sqrt{z^2+a^2}}{c}$. Also, $I(t) = I_0\theta(t)$ and $\dot{I}(t) = I_0\delta(t)$ so we get

$$B_3(\vec{x}, t) = \frac{\mu_0}{4\pi}\int \Big[\frac{I_0}{z^2+a^2}\theta\Big(t - \frac{\sqrt{z^2+a^2}}{c}\Big) + \frac{I_0}{c\sqrt{z^2+a^2}}\delta\Big(t - \frac{\sqrt{z^2+a^2}}{c}\Big)\Big](\vec{dl'} \times \hat{e}_R)_3$$

$$= \frac{\mu_0 I_0}{4\pi(z^2+a^2)}\theta\Big(t - \frac{\sqrt{z^2+a^2}}{c}\Big)\int(\vec{dl'} \times \hat{e}_R)_3 + \frac{\mu_0 I_0}{4\pi c\sqrt{z^2+a^2}}\delta\Big(t - \frac{\sqrt{z^2+a^2}}{c}\Big)\int(\vec{dl'} \times \hat{e}_R)_3$$

From geometry

$$(\vec{dl'} \times \hat{e}_R)_3 \;=\; dl'\frac{a}{\sqrt{a^2+z^2}} \qquad\Rightarrow\qquad \int(\vec{dl'} \times \hat{e}_R)_3 \;=\; \frac{2\pi a^2}{\sqrt{z^2+a^2}}$$

and therefore

$$B(z,t) \;=\; \frac{\mu_0 I_0 a^2 \hat{e}_3}{2(z^2+a^2)^{3/2}}\theta\Big(t-\frac{\sqrt{z^2+a^2}}{c}\Big)+\frac{\mu_0 I_0 a^2 \hat{e}_3}{2c(z^2+a^2)}\delta\Big(t-\frac{\sqrt{z^2+a^2}}{c}\Big)$$

Note that the first term is equal to the result for constant current I_0 multiplied by $\theta\Big(t-\frac{\sqrt{z^2+a^2}}{c}\Big)$ (see e.g. Eq. (5.38) from Griffiths textbook). The second part is a burst of radiation coming from turning on the current.

Problem 3.

Find the reflection coefficient for the circularly polarized electromagnetic wave incident on a plane between two linear media at Brewster's angle. (For simplicity, take $\mu' = \mu$).

Solution:

The circularly polarized wave have the form

$$\vec{E} = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2)E_0^i e^{ik(z\cos\theta_i + x\sin\theta_i)}$$

(we've assumed that $\vec{k}$ lies in the XZ plane).

Let us consider the components $E_0^i\hat{e}_1$ (electric field in the plane of incidence) and $iE_0^i\hat{e}_2$ (electric field normal to the plane of incidence) separately.

At $\theta_i = \theta_B$ we have no reflected wave for the component $E_0^i\hat{e}_1$ (in the plane of incidence). For $iE_0^i\hat{e}_2$ (normal to the plane of incidence) we have

$$E_0^r = \Big(\frac{i}{\sqrt{2}}E_0^i\Big)\frac{\sin(\theta_T-\theta_i)}{\sin(\theta_T+\theta_i)}$$

Since at $\theta_i = \theta_B$ we have $\sin\theta_B = \cos\theta_T = n'/\sqrt{n^2+n'^2}$ and $\cos\theta_B = \sin\theta_T = n/\sqrt{n^2+n'^2}$, we obtain

$$E_0^r = \frac{i}{\sqrt{2}}E_0^i\frac{\sin(\theta_T-\theta_B)}{\sin(\theta_T+\theta_B)} = \frac{i}{\sqrt{2}}E_0^i\frac{n^2-n'^2}{n^2+n'^2}$$

Therefore, the intensity of the reflected wave is proportional to $\frac{1}{2}\Big(E_0^i\frac{n^2-n'^2}{n^2+n'^2}\Big)^2$ and the reflection coefficient is

$$R = \frac{(n^2-n'^2)^2}{2(n^2+n'^2)^2}$$